Quantum Field Theory II — table of contents:

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Quantum Field Theory II :

Path integrals and Renormalization

* Goal of the course: introduce the path integral approach to QFT, which is an elegant formulation compactly encoding all results obtained through canonical quantization and Feynman rules. It can also lead to define the most general QFTs. Using this approach, we will discuss symmetries, radiative correction and the ensuing renormalization, eventually leading to the phenomenon of coupling constant evolution (along the energy scale).

1 * We start with the path integral formulation of quantum mechanics (QM):

Consider a system with a coordinate q, a conjugate momentum p and a Hamiltonian $H(p,q)$.

We will mostly take $H(p,q) = \frac{1}{2m}p^2 + V(q)$

but the argument can be easily generalized to a more general H.

We can consider the evolution of the wave function as:

$$i\hbar \frac{\partial}{\partial t} |q(t)\rangle = \hat{H} \, |q(t)\rangle$$

where we take here the Schrödinger picture where the wave function evolves in time; $\hat{H}$ is the Hamiltonian as an operator, where p and q have been replaced by their quantum operators $\hat{P}$ and $\hat{Q}$.

Let us write, fixing a reference time t_0:

$$|q(t)\rangle = U(t, t_0) \, |q(t_0)\rangle$$

$U(t, t_0)$ is the evolution operator, and it satisfies the same Schrödinger eq.

If H does <u>not</u> depend explicitly on time, which we will assume from now on, then U has a simple expression:

$$U(t, t_0) = e^{-\frac{i}{\hbar} H \cdot (t - t_0)}$$ we set $\hbar = 1$ hereafter.

• Before computing amplitudes for transitions from q to q' in a fixed time, let us go back to the operators $\hat{P}$ and $\hat{Q}$:

unfortunately, $[\hat{Q}, \hat{P}] = i$

We cannot diagonalize $\hat{P}$ and $\hat{Q}$ simultaneously

→ we define 2 complete sets of eigenstates:

$\hat{Q}|q\rangle = q|q\rangle$, $\hat{P}|p\rangle = p|p\rangle$

complete, orthonormal:

$\langle q'|q\rangle = \delta(q-q')$ $\quad\quad \mathbb{1} = \int dq\, |q\rangle\langle q|$

$\langle p'|p\rangle = \delta(p'-p)$ $\quad\quad \mathbb{1} = \int dp\, |p\rangle\langle p|$

What about $\langle q|p\rangle$?

Recall that on a state $|\psi\rangle$, $\langle q|\psi\rangle = \psi(q)$ is the wave function in terms of q.

On such wave function

$\langle q|\hat{P}|\psi\rangle = -i\frac{\partial}{\partial q}\psi(q)$

so that $[\hat{Q},\hat{P}] = i$ is satisfied

Then: $\langle q|\hat{P}|p\rangle = -i\frac{\partial}{\partial q}\langle q|p\rangle = p\langle q|p\rangle$

$\Rightarrow \quad \langle q|p\rangle = \alpha\, e^{ipq}$

Normalization:

$\langle q'|q\rangle = \delta(q'-q) = \int dp\, \langle q'|p\rangle\langle p|q\rangle$

$= \int dp\, |\alpha|^2 e^{ipq'} e^{-ipq} = |\alpha|^2 \int dp\, e^{ip(q'-q)}$

$= |\alpha|^2\, 2\pi\, \delta(q'-q) \quad\quad \Rightarrow \quad \alpha = \frac{1}{\sqrt{2\pi}}$.

Eventually : $$\boxed{\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\, e^{ipq}}$$

* We now want to evaluate the amplitude of going from q at time t_0 to q' at time t' :

$$\langle q' | U(t',t_0) | q\rangle$$

Let us first take an infinitesimal time step :

$t' = t + dt$

$$\langle q' | U(t+dt,t) | q\rangle = \langle q' | e^{-i\hat{H}dt} | q\rangle$$

$$\simeq \langle q' | (\mathbb{1} - i\hat{H}dt) | q\rangle$$

Now, generically we have $\hat{H}(\hat{P},\hat{Q})$ → if there is ordering ambiguity, put all $\hat{P}$'s on the right.

We then insert a complete $|p\rangle$ basis :

$$\langle q' | U(t+dt) | q\rangle = \int dp\, \langle q' | (\mathbb{1} - i\hat{H}\, dt) | p\rangle \langle p | q\rangle$$

$$= \int dp\, (1 - iH(p,q')dt)\, \langle q' | p\rangle \langle p | q\rangle$$

$$\simeq \int \frac{dp}{2\pi}\, e^{ip(q'-q) - iH(p,q')dt}$$

Now, for the finite time evolution, we cut the time interval in N infinitesimal pieces, $N \to \infty$.

And of course, we have to integrate over any position at each intermediate infinitesimal step:

$$\langle q'|U(t',t)|q\rangle = \lim_{N\to\infty} \langle q'|U(t',t_N)U(t_N,t_{N-1})\dots U(t_2,t_1)U(t_1,t)|q\rangle$$

$$= \lim_{N\to\infty}\int dq_N \dots dq_1 \langle q'|U(t',t_N)|q_N\rangle\langle q_N|U(t_N,t_{N-1})|q_{N-1}\rangle \dots$$

$$\dots \langle q_2|U(t_2,t_1)|q_1\rangle\langle q_1|U(t_1,t)|q\rangle$$

considering each $t_k - t_{k-1} = dt$ infinitesimal, we can substitute:

$$\langle q_k|U(t_k,t_{k-1})|q_{k-1}\rangle = \int \frac{dp_k}{2\pi}\, e^{ip_k(q_k-q_{k-1}) - iH(p_k,q_k)dt}$$

so that:

$$\langle q'|U(t',t)|q\rangle = \int \prod_{k=1}^{N} dq_k \prod_{k=1}^{N+1}\frac{dp_k}{2\pi}\, e^{ip_{N+1}(q'-q_N) - iH(p_{N+1},q')dt + \sum_{k=1}^{N}\left[ip_k(q_k-q_{k-1}) - iH(p_k,q_k)dt\right]}$$

one is really integrating over all paths from q to q'

Let us go to the continuum limit:

$q_k = q(t_k)$ $\quad p_k = p(t_k)$ $\quad$ let us also define $t_0 \equiv t$ $\quad t_{N+1} = t'$.

$$\langle q'|U(t',t)|q\rangle = \int \prod_{k=1}^{N} dq(t_k) \prod_{k=1}^{N+1} \frac{dp(t_k)}{2\pi}\, e^{i\sum_{k=1}^{N+1}\left[p(t_k)\left(q(t_k)-q(t_{k-1})\right) - H(p(t_k),q(t_k))dt\right]}$$

$$= \int \prod_{k=1}^{N} dq(t_k) \prod_{k=1}^{N+1} \frac{dp(t_k)}{2\pi}\, e^{i\sum_{k=1}^{N+1} dt\left[p(t_k)\dot{q}(t_k) - H(p(t_k),q(t_k))\right]}$$

$$= \int_{\substack{q(t)=q \\ q(t')=q'}} \mathcal{D}q(\tau)\,\mathcal{D}\frac{p(\tau)}{2\pi}\, e^{i\int_t^{t'} d\tau\left[p(\tau)\dot{q}(\tau) - H(p(\tau),q(\tau))\right]}$$

$$= \int \mathcal{D}q\,\mathcal{D}\frac{p}{2\pi}\, e^{iS_H(p,q)}$$

$S_H = \int dt\,(p\dot{q} - H(p,q))$ is the Hamiltonian <u>action</u>

* Now, consider an Hamiltonian which is "canonical".

$$H = \frac{1}{2m}p^2 + V(q)$$

It can be generalized, what is important is that H is at most quadratic in p. Then, the integrals over dp are essentially Gaussian integrals:

$$\int \frac{dp}{2\pi}\, e^{ip\left(\frac{dq}{dt}dt\right) - i\frac{1}{2m}p^2dt - iVdt} = \int \frac{dp}{2\pi}\, e^{-i\frac{dt}{2m}\left(p-m\frac{dq}{dt}\right)^2 + \frac{i}{2}mdt\,\dot{q}^2 - iVdt}$$

$$= e^{i\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)dt} \int \frac{d\tilde{p}}{2\pi}\, e^{-\frac{i\,dt}{2m}\tilde{p}^2}$$

The Gaussian integral is:

$$\int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2} = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{+\infty} dz\, e^{-z^2} = \frac{1}{\sqrt{\alpha}} \left(\int_{\mathbb{R}^2} dx\,dy\, e^{-x^2-y^2} \right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\alpha}} \left(2\pi \int_0^{\infty} dr\, r\, e^{-r^2} \right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{\alpha}}$$

Thus eventually,

$$\langle q+dq | U(t+dt,t) | q \rangle = \frac{1}{2\pi} \sqrt{\frac{2\pi m}{i\,dt}}\, e^{i\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)dt} = \sqrt{\frac{m}{2\pi i\,dt}}\, e^{iL(q)dt}$$

The Lagrangian appears naturally!

The normalization factor will be irrelevant, so we will just keep in mind it exists.

For the finite amplitude, performing each $\int dp_n$ one gets eventually:

$$\langle q' | U(t',t) | q \rangle = N \int \mathcal{D}q(\tau)\, e^{i\int_t^{t'} d\tau \left(\frac{1}{2}m\dot{q}(\tau)^2 - V(q(\tau))\right)} = N \int \mathcal{D}q(\tau)\, e^{iS_L}$$

where $S_L = \int d\tau\, L(q,\dot{q}) = \int d\tau \left(\frac{1}{2}m\dot{q}^2 - V(q) \right)$

is the (Lagrangian) action

* Note that we could get to this expression performing the Gaussian path integral $\int \mathcal{D}p(\tau)$. Indeed:

$$\int \mathcal{D}q(\tau)\mathcal{D}p(\tau)\, e^{i\int d\tau\left(p\dot{q} - \frac{1}{2m}p^2 - V(q)\right)} =$$

$$= \int \mathcal{D}q\mathcal{D}p\, e^{-\frac{i}{2m}\int d\tau (p - m\dot{q})^2 + i\int d\tau\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)}$$

$$= \int \mathcal{D}q\, e^{i\int d\tau\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)} \underbrace{\int \mathcal{D}\tilde{p}\, e^{-\frac{i}{2m}\int d\tau\, \tilde{p}^2}}_{N}$$

We will from now on, mostly consider the Lagrangian path int.

* What happens if we insert some operators under the path integral? E.g.

$$\int \mathcal{D}q\, e^{iS}\, O_1(\bar{t}_1)\, O_2(\bar{t}_2)$$ assuming, obviously, $t < \bar{t}_1, \bar{t}_2 < t'$

if, say, $t_1 < t_2$, we can think again of the discretization of the path integral and write

$$\int \mathcal{D}q\, e^{iS}\, O_1(\bar{t}_1)\, O_2(\bar{t}_2) = \int \prod_{k=1}^{N} dq(t_k)\, e^{i\sum_{k=k_2}^{N}\left(\frac{1}{2}m\dot{q}_k^2 - V(q_k)\right)}\, O_2(\bar{t}_2)\, e^{i\sum_{k=k_1}^{k_2-1}\left(\frac{1}{2}m\dot{q}_k^2 - V(q_k)\right)}$$

$$\times O_1(\bar{t}_1)\, e^{i\sum_{k=1}^{k_1-1}\left(\frac{1}{2}m\dot{q}_k^2 - V(q_k)\right)}$$

where $t_{k_1-1} < \bar{t}_1 < t_{k_1}$ and $t_{k_2-1} < \bar{t}_2 < t_{k_2}$

This is proportional to

$$\langle q' | U(t',t_2) O_2(t_2) U(t_2,t_1) O_1(t_1) U(t_1,t) | q \rangle$$

Had we chosen $t_2 < t_1$, we would interchange the roles of O_1 and O_2 and get

$$\langle q' | U(t',t_1) O_1(t_1) U(t_1,t_2) O_2(t_2) U(t_2,t) | q \rangle .$$

The path integral thus gives always the time-ordered correlation function of the various operators.
Symbolically, and tacitly assuming the evolution operators U, it is usually written as:

$$\langle q' | T(O_1(t_1) O_2(t_2) \ldots) | q \rangle = N \int \mathcal{D}q \, (O_1(t_1) O_2(t_2) \ldots) e^{iS(q)}$$

Let us now go to field theory:

First, generalize the above to a large number of degrees of freedom, n : q^i

The path integral becomes

$$\int \mathcal{D}q^i(t) \, e^{i \int d\tau \sum_i \left(\frac{1}{2} m \dot{q}_i^2 - V(q_i) \right)}$$

The transition from QM to QFT is made taking a continuous index $i \to x$ a dynamical variable at every space point.

→ $q \to \phi$ relabel, and implement Poincaré invariance.

we eventually get to

$$S(\phi) = \int d^4x\, \mathcal{L}(\phi) \qquad \mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

Note that from a Hamiltonian pt of view, we have:

$$S = \int dt \int d^3x \left\{ \frac{1}{2}\dot{\phi}^2 \underbrace{- \frac{1}{2}(\partial_i \phi)^2 - V(\phi)}_{\text{"}-V(q)\text{"}} \right\}$$

of before

⚠ mostly minus metric $\eta = (+ - - -)$

The path integral is then

$$\langle \phi_f(\vec{x}) | U(T,-T) | \phi_i(\vec{x}) \rangle = \int \mathcal{D}\phi(x)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)}$$

with the conditions
$$\phi(-T,\vec{x}) = \phi_i(\vec{x})$$
$$\phi(T,\vec{x}) = \phi_f(\vec{x}) .$$

This expression is manifestly Poincaré invariant, except for the boundary conditions

→ we will take this as the <u>definition</u> of a QFT, from the path integral over field configurations, weighted by ~~kng~~ the phase given by the (Lagrangian) action.

(2) Path integral for a scalar field theory.

We consider the theory defined by the path integral

$$\int \mathcal{D}\phi(x)\, e^{iS(\phi)}$$

and see how to use it to define correlation functions of operators, including the field ϕ itself. (we consider a real scalar field for the moment).

For instance, we would like to compute Green functions such as

$$\langle 0|\, \phi(x_1)\, \phi(x_2)\, |0\rangle \qquad \text{or with more fields.}$$

Note that we are usually interested in vacuum amplitudes, and without explicit boundary conditions at (large) times T and $-T$. The two issue turn out to be related.

Take for the moment $\quad S = \int_{-T}^{T} d^4x\, \mathcal{L}(\phi)$

and boundary conditions $\phi(-T,\vec{x}) = \phi_i(\vec{x})$, $\phi(T,\vec{x}) = \phi_f(\vec{x})$

$$\int \mathcal{D}\phi\ \phi(x_1)\,\phi(x_2)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}$$

can be computed as follows: suppose $x_1^0 < x_2^0$

and call $\phi_1(\vec{x}) = \phi(x_1^0,\vec{x})$, $\phi_2(\vec{x}) = \phi(x_2^0,\vec{x})$

then we can split the path integral in 3:

$$\int \mathcal{D}\phi \; e^{i\int_{-T}^{x_1^0} d^4x \mathcal{L}} \; \phi_1(\vec{x}) \int \mathcal{D}\phi \; e^{i\int_{x_1^0}^{x_2^0} d^4x \mathcal{L}} \; \phi_2(\vec{x}) \int \mathcal{D}\phi \; e^{i\int_{x_2^0}^{T} d^4x \mathcal{L}}$$

$$= \langle \phi_f | U(T, x_2^0) \, \phi(\vec{x}_2) \, U(x_2^0, x_1^0) \, \phi(\vec{x}_1) \, U(x_1^0, -T) | \phi_i \rangle$$

Turning to Heisenberg picture operators (at a reference time, taken for convenience at $x^0 = 0$), and recalling that an equivalent expression with $\phi(x_1)$ and $\phi(x_2)$ interchanged is obtained if $x_1^0 > x_2^0$, we get

$$\int \mathcal{D}\phi \; \phi(x_1) \, \phi(x_2) \, e^{i\int_{-T}^{T} d^4x \mathcal{L}} = \langle \phi_f | e^{-iHT} \, T\left(\phi(x_1)\,\phi(x_2)\right) e^{-iHT} | \phi_i \rangle$$

we would like now to take the limit $T \to \infty$, pushing the boundary conditions to the infinite past & future.

Take then, for instance:

$$e^{-iHT} | \phi_i \rangle = \sum_{|n\rangle} e^{-iE_n T} | n \rangle \langle n | \phi_i \rangle$$

$$= e^{-iE_0 T} | 0 \rangle \langle 0 | \phi_i \rangle + \sum_{n \neq 0} e^{-iE_n T} | n \rangle \langle n | \phi_i \rangle$$

Now, since $E_n > E_0$ by definition ($|0\rangle$ is vacuum) if we take $T \to \infty(1 - i\varepsilon)$ then all but the first term will disappear, provided of course $\langle 0 | \phi_i \rangle \neq 0$, which we assume otherwise ϕ_i is a state that cannot be produced from the vacuum.

Thus we obtain that

$$\lim_{T\to\infty(1-i\varepsilon)} \int \mathcal{D}\phi \, (\phi(x_1)\ldots)\, e^{i\int_{-T}^{T} d^4x\,\mathcal{L}} = \left(e^{-2iE_0T} \langle 0|\phi_i\rangle \langle \phi_f|0\rangle\right) \langle 0|T\{\phi(x_1)\ldots\}|0\rangle$$

if we normalize $\langle 0|0\rangle = 1$, we finally get:

$$\boxed{\langle 0|T\{\phi(x_1)\ldots\}|0\rangle = \frac{\int \mathcal{D}\phi\,(\phi(x_1)\ldots)\, e^{i\int d^4x\,\mathcal{L}}}{\int \mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}}}}$$

where $S = \int d^4x\,\mathcal{L}$ is now integrated over all of spacetime.

* We will now derive propagators and Feynman rules from this path integral definition of the QFT, rather than from canonical quantization.
The key point is to start from a free theory. We will then introduce interaction as perturbations of the free theory. (Note however that, in principle, the above expression defines the correlator of a QFT even outside the perturbative regime. For instance, one can put the QFT on a lattice and numerically compute exact correlators.)

For a free scalar theory, the basic path integral is

$$\int \mathcal{D}\phi \, (\phi(x_i) \ldots) \, e^{i\int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2\right)}$$

The phase is quadratic in ϕ, so the path integral is essentially a Gaussian integral. We can perform it, in the same way as we did the $\mathcal{D}p(\tau)$ integral before. But we have to take care of the $\phi(x_i)$ insertions also. Let us remind ourselves of the "discrete" usual Gaussian integrals:

without insertions, we have $(i = 1 \ldots n)$

$$\int dv_i \, e^{-v_i k_{ij} v_j} = \int dv_i' \, e^{-v_i' k_i v_i'}$$

$$= \frac{\pi^{n/2}}{\prod_{i=1}^{n} k_i^{1/2}} = \sqrt{\frac{\pi^n}{\det K}}$$

v_i' basis that diagonalizes k_{ij}, e.v. k_i

$v_i \overset{n}{=} O_{ij} v_j'$ O_{ij} orthogonal.

with insertions:

$$\int dv_i \, v_k v_\ell \, e^{-v_i k_{ij} v_j} = \int dv_i' \, O_{km} v_m' \, O_{\ell n} v_n' \, e^{-v_i' k_i v_i'}$$

$$= \sum_m O_{km} O_{\ell m} \sqrt{\frac{\pi^n}{\det k}} \, \frac{1}{2k_m}$$

the complicated $\frac{1}{\sqrt{\det k}}$ factor cancels from the ratio

$$\frac{\int (dv)\, v^2 e^{-vkv}}{\int (dv)\, e^{-vkv}}$$ and the $\frac{1}{k}$ behavior can be guessed

by observing that $[v^2] = [k]^{-1}$ dimensionally.

* In order to compute effortlessly the n-pt functions of the free theory, we will use a trick: write the generating function of all such correlators.

One introduces a source for ϕ, an external field such that, deriving with respect to it, one creates the insertions under the path integral.

Define

$$Z[J] = \int \mathcal{D}\phi \; e^{i\int d^4x \,(\mathcal{L}(\phi) + J(x)\phi(x))}$$

$J(x)$ is an external field, hence Z depends on it.

Define also the functional derivative (same as the one used to compute Euler-Lagrange equations):

$$\frac{\delta}{\delta J(x)} J(y) = \delta^4(x-y)$$

$$\rightarrow \quad \frac{\delta}{\delta J(x)} \int d^4y \, J(y)\phi(y) = \phi(x)$$

$$\frac{\delta}{\delta J(x)} \, e^{i\int d^4y \, J(y)\phi(y)} = i\,\phi(x)\, e^{i\int d^4y\, J(y)\phi(y)}$$

Then we have

$$\frac{\delta}{\delta i J(x)} Z[J] = \int \mathcal{D}\phi \, \phi(x) \, e^{i\int d^4x (\mathcal{L}(\phi) + J\phi)}$$

one can take as many derivatives one wants:

$$\left(\frac{\delta}{\delta i J(x_1)} \frac{\delta}{\delta i J(x_2)} \cdots \right) Z[J] = \int \mathcal{D}\phi \left(\phi(x_1) \phi(x_2) \ldots \right) e^{i\int d^4x (\mathcal{L} + J\phi)}$$

If, after taking the $\frac{\delta}{\delta J}$ derivatives, we set $J=0$, we get back the path integrals we discussed before. Hence, once $Z[J]$ is defined, we have

$$\langle 0 | T\{ \phi(x_1)\phi(x_2) \ldots \} | 0 \rangle = \frac{1}{Z[J]} \left(\frac{\delta}{\delta i J(x_1)} \frac{\delta}{\delta i J(x_2)} \cdots \right) Z[J] \Big|_{J=0}$$

* Now, $Z(J)$ is just a Gaussian integral with a shift
→ it can be easily computed:

$$\int d^4x (\mathcal{L} + J\phi) = \int d^4x \left\{ \frac{1}{2}\phi(-\Box - m^2)\phi + J\phi \right\}$$

in order to complete the square, we need to shift ϕ:

$$\frac{1}{2}\phi k \phi + J\phi = \frac{1}{2}(\phi + \cancel{k^{-1}} J k^{-1}) k (\phi + k^{-1}J) - \frac{1}{2} J k^{-1} J$$

$k \equiv -\Box - m^2$ so that k^{-1} is such that

$$(-\Box - m^2)\, k^{-1} = \delta^4(x-y)$$

$\to$ k^{-1} is the two-pt Green function!

Let us call it D. If we redefine $\phi' = \phi + \int D J$

$$Z(J) = e^{-\frac{1}{2} i \int d^4x\, d^4y\, J(x) D(x-y) J(y)} \int \mathcal{D}\phi' \, e^{i\int d^4x\, \phi'(-\Box - m^2 + i\varepsilon)\phi'}$$

Note the $+i\varepsilon$ prescription in order to have indeed a convergent Gaussian integral.

The actual integral over ϕ' goes entirely in the normalization of Z, which cancels from the Green functions.

* The 2-pt function, i.e. the propagator, is thus

$$\langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle = \frac{1}{Z} \frac{\delta}{\delta i J(x)} \frac{\delta}{\delta i J(y)} Z(J) \Big|_{J=0} = i\, D(x-y)$$

$$= \frac{i}{-\Box - m^2 + i\varepsilon}\, \delta^4(x-y)$$

Feynman propagator

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$\hookrightarrow$ in Fourier space.

* Higher pt. functions can be compute as straightforwardly.
  E.g. the 4-pt function:

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle \equiv \langle \phi_1\phi_2\phi_3\phi_4\rangle \quad (\text{shorthand!})$$

$$= \frac{\delta}{\delta iJ_1}\frac{\delta}{\delta iJ_2}\frac{\delta}{\delta iJ_3}\frac{\delta}{\delta iJ_4}\, e^{-\frac{1}{2}i\int J_x D_{xy} J_y}$$

$$= \frac{\delta}{\delta iJ_1}\frac{\delta}{\delta iJ_2}\frac{\delta}{\delta iJ_3}\left(-\int D_{x4} J_x\, e^{-\frac{1}{2}i\int J_x D_{xy} J_y}\right)$$

$$= \frac{\delta}{\delta iJ_1}\frac{\delta}{\delta iJ_2}\left(iD_{34}\, e^{-\frac{1}{2}i\int J_x D_{xy} J_y} + \int D_{x4} J_x \int D_{y3} J_y\, e^{-\frac{1}{2}\int iJ_x D_{xy} J_y}\right)$$

$$= \frac{\delta}{\delta iJ_1}\left\{ iD_{34}\left(-\int D_{x2} J_x\right) - iD_{24}\int D_{y3} J_y - iD_{23}\int D_{x4} J_x \right\} e^{-\frac{1}{2}i\int J_x D_{xy} J_y}$$

$$= iD_{12}\, iD_{34} + iD_{13}\, iD_{24} + iD_{14}\, iD_{23}$$

→ the path integral gives us automatically all the Wick contractions!

* We can now go beyond the free theory, and add interactions. In our real scalar case, the simplest term one can write is the quartic one:

$$\mathcal{L} \to \underbrace{\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2}_{\mathcal{L}_0} - \underbrace{\frac{1}{4}\lambda\phi^4}_{\mathcal{L}_{int}}$$

The path integral of correlation functions now reads

$$\int \mathcal{D}\phi \, (\phi(x_1)\phi(x_2)\ldots)\, e^{i\int \mathcal{L}_0 + i\int \mathcal{L}_{int}}$$

$\mathcal{L}_{int}$ can be treated as a perturbation:

$$\cancel{\int\mathcal{D}\phi}\ \langle \cancel{\phi}\,\phi(x_1)\,\phi(x_2)\ldots\rangle_{int} = \int \mathcal{D}\phi \, (\phi(x_1)\phi(x_2)\ldots)\left(1 + i\int \mathcal{L}_{int}(\phi) + \ldots\right) e^{i\int \mathcal{L}_0}$$

$$= \langle \phi_1 \phi_2 \ldots \rangle_{free} + i \,\langle (\phi_1\phi_2\ldots)\int \mathcal{L}_{int}(\phi)\rangle_{free} + \ldots$$

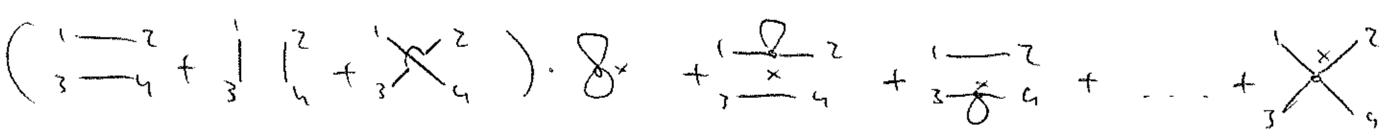
The correlators in the interacting theory can be computed in the free theory, in an expansion (in λ)

For instance, let us compute the first correction to the free result for the 4-pt function:

$$\langle \phi_1\phi_2\phi_3\phi_4\rangle_{int} = \langle \phi_1\phi_2\phi_3\phi_4\rangle_{free} + \frac{i\lambda}{4}\langle \phi_1\phi_2\phi_3\phi_4 \int d^4x\, \phi_x^4 \rangle_{free}$$

by the same rules as before, we see that we can contract in many ways $\langle \phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x\rangle$

Pictorially:



The only connected graph gives:

$$-\frac{i\lambda}{4}\cdot 4! \int d^4x \; iD_{1x}\, iD_{2x}\, iD_{3x}\, iD_{4x} = (-6i\lambda)\int d^4x \; iD_{1x}\ldots$$

These are the usual Feynman rules!

And Feynman diagrams of course.

We have thus already encountered 2 main features in these diagrams: there are connected and disconnected ones; there are ones with loops.

- The ~~ones~~ disconnected ones are redundant — we will get rid of them.
- The ones with loops are higher order corrections to ones without loops: E.g.

$$\langle \phi_1 \phi_2 \rangle_{int} = \langle \phi_1 \phi_2 \rangle_{free} + i \int d^4x \, \langle \phi_1 \phi_2 \mathcal{L}_{int} \rangle + \ldots$$

the correction is

$$-i\frac{\lambda}{4}\int d^4x \, \langle \phi_1 \phi_2 \phi_x \phi_x \phi_x \phi_x \rangle = -3i\lambda \int d^4x \; iD_{1x} \, iD_{2x} \, iD_{xx} + disc.$$

This is an $O(\lambda)$ correction to the iD_{12} propagator (which obviously does not depend on λ).

(We might already suspect that there is something ill-defined by the presence of the iD_{xx} propagator in the internal line $\rightarrow$ this will be the subject of renormalization.)

The complete (exact, physical) propagator will be the one where all corrections have been summed.

* In order to get rid of disconnected diagrams, let us go back to $Z[J]$ in the free theory.
Here the only connected diagram is the 2-pt function propagator, all higher pt (even) functions are disconnected.

Now $\quad Z[J] = e^{-i\frac{1}{2}\int JDJ}$

What generates the (only) connected diagram is manifestly $\log Z[J]$.

Let us define $W[J]$ such that $Z[J] = e^{-iW[J]}$

Then, in the free theory, $W[J]$ generates only the connected Green functions:

$$\langle \phi_1 \phi_2 \rangle = \frac{\delta}{\delta iJ_1} \frac{\delta}{\delta iJ_2} \left(-iW(J) \right) = iD_{12}$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \frac{\delta}{\delta iJ_1} \frac{\delta}{\delta iJ_2} \frac{\delta}{\delta iJ_3} \frac{\delta}{\delta iJ_4} \left(-i\frac{1}{2} \int J_x D_{xy} J_y \right) = 0$$

We simply extend this definition to the interacting theory, or to QFTs in general:

$$e^{-iW(J)} = \int \mathcal{D}\phi \, e^{iS_{tot} + i\int J\phi}$$

* Consider for simplicity our $\lambda\phi^4$ theory, which now contains connected higher pt. functions.
It has though a $\mathbb{Z}_2$ symmetry $\phi \to -\phi$ that sets to zero all odd-pt functions.

Then, assuming $W(J)$ indeed generates connected Green functions, we can write it as

$$-iW(J) = \frac{1}{2}\int d^4x\, d^4y \; iJ(x)\, iJ(y) \;\langle \phi(x)\phi(y)\rangle_c$$

$$+ \frac{1}{4!}\int d^4x_1 \ldots d^4x_4 \; iJ(x_1)\, iJ(x_2)\, iJ(x_3)\, iJ(x_4) \;\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_c$$

$$+ \ldots$$

we can then compute $Z(J) = e^{-iW(J)}$

$$= 1 - iW(J) + \frac{1}{2}\left(-iW(J)\right)^2 + \ldots$$

so that

$$Z(J) = 1 + \frac{1}{2}\iint iJ_1\, iJ_2 \;\langle \phi_1\phi_2\rangle_c + \frac{1}{4!}\iiiint iJ_1\, iJ_2\, iJ_3\, iJ_4 \;\langle \phi_1\phi_2\phi_3\phi_4\rangle_c$$

$$+ \frac{1}{8}\iiiint iJ_1\, iJ_2\, iJ_3\, iJ_4 \;\langle \phi_1\phi_2\rangle_c \langle \phi_3\phi_4\rangle_c$$

$$+ \mathcal{O}(J^6)$$

From this we see that

$$\frac{\delta}{\delta iJ_1}\frac{\delta}{\delta iJ_2}\frac{\delta}{\delta iJ_3}\frac{\delta}{\delta iJ_4} Z(J) = \langle \phi_1\phi_2\phi_3\phi_4\rangle$$

$$= \langle \phi_1\phi_2\phi_3\phi_4\rangle_c + \langle \phi_1\phi_2\rangle_c \langle \phi_3\phi_4\rangle_c$$

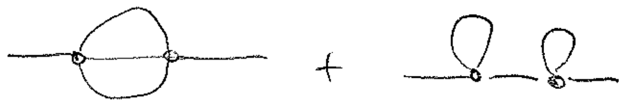
$$+ \langle \phi_1\phi_3\rangle_c \langle \phi_2\phi_4\rangle_c + \langle \phi_1\phi_4\rangle_c \langle \phi_2\phi_3\rangle_c .$$

indeed sum of connected + disconnected diagrams, with correct prefactors

(we have neglected / put to 1 the vacuum diagrams)

- The same constructive proof can be shown for a more general theory without $\mathbb{Z}_2$ symmetry, where also odd-pt functions (1-pt, 3-pt, ...) are non trivial, with the same result. A general proof by induction can also be given.

* We have not removed yet all redundancy from the order by order computation of correlators.
  For instance, still in $\lambda\phi^4$ theory, we can consider the $O(\lambda^2)$ corrections to the propagator:



only 2 connected diagrams.

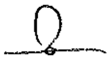
Obviously, the second does not contain anything new ~~from~~ with respect to the $O(\lambda)$ diagram 
It is just a repetition/gluing of 2 $O(\lambda)$ diagrams, joined by an internal line. ~~which~~ Breaking this internal line in 2, one obtains the 2 separate $O(\lambda)$ diagrams. Diagrams which <u>cannot</u> be split in 2 in this way (cutting an internal propagator) are called <u>one-particle irreducible</u> (1PI).
They are the only "new" ones at every order in perturbation theory.

8 Take indeed the example of the propagator in $\lambda\phi^4$:

$\langle \phi(x)\phi(y)\rangle = iD(x-y)$ in the free theory

in Fourier space $iD = \frac{i}{k^2-m^2}$ (assume $+i\varepsilon$ prescription)

The $O(\lambda)$ correction is (schematically)

 $iD(-iM)\,iD$ where $-iM$ is the (presumably divergent/problematic) correction.

At $O(\lambda^2)$, this correction repeat itself as:

 $iD(-iM)\,iD(-iM)\,iD$

and so on; this sequence of non-1PI diagrams can be easily summarized as

$$iD_{sum} = iD\sum_{n=0}^{\infty}(-iM)^n(iD)^n = iD\sum_{n=0}^{\infty}(MD)^n$$

it can be summed explicitly: it is a geometric series:

$$iD_{sum} = \frac{iD}{1-MD} = \frac{i}{D^{-1}-M} = \frac{i}{k^2-m^2-M(k^2,m^2)}$$

What really matters, to compute the corrected propagator, is to compute $-iM(k^2,m^2)$ to the desired order in λ.

$-iM \equiv$  "amputated" diagram $\equiv$ do not take into account external lines.

* These will be the corrections that we will really compute. There is a generating functional also for them, the <u>effective action</u>.

Intuitively, indeed something like $-iM$ seems like a correction directly to the action: in Fourier space

$$S = \int \frac{d^4k}{(2\pi)^4} \phi(-k)^* (k^2-m^2)\phi(k) \to \int \frac{d^4k}{(2\pi)^4} \phi(-k)^* (k^2-m^2-M(k^2,m^2))\phi(k)$$

we need an object, similar to the action, that yields for the 2 pt function, $D^{-1}_{sum} = D^{-1}_{free} - M_{1PI}$

Consider

$$e^{-iW(J)} = \int \mathcal{D}\phi \, e^{iS + i\int J\phi}$$

in some way, $-iW$ is iS to which $iJ\phi$ is added, like the free energy in thermodynamics.
To get back to the "action" we should just perform a Legendre transform.
Let us call

$$\phi_{cl} = \frac{\delta}{\delta iJ}(-iW(J)) \equiv \langle \phi \rangle_J \quad \text{at } \underline{\text{sources on}}$$

(we do <u>not</u> set $J=0$ after $\frac{\delta}{\delta iJ}$)

ϕ_{cl} depends on J, and the relation can be inverted so as to express J in terms of ϕ_{cl}.

Then we define the effective action (traditionally labelled Γ) as

$$i\Gamma(\phi_{cl}) = -iW(J) - i\int J\,\phi_{cl}$$

where J is expressed in terms of ϕ_{cl}.

It is easy to see that (as usual for Legendre transforms)

$$\frac{\delta}{\delta\phi_{cl}}\Gamma = \frac{\delta}{\delta i\phi_{cl}}i\Gamma = \frac{\delta}{\delta i\phi_{cl}}(-iW) - \int\frac{\delta J}{\delta\phi_{cl}}\phi_{cl} - J$$

$$= \int \underbrace{\frac{\delta J}{\delta\phi_{cl}}\frac{\delta}{\delta iJ}(-iW)}_{\text{chain rule}} - \int\frac{\delta J}{\delta\phi_{cl}}\underbrace{\frac{\delta}{\delta iJ}(-iW)}_{\text{def of } \phi_{cl}} - J$$

$$= -J$$

This relation implies

$$\delta^4(x-y) = \frac{\delta}{\delta J(x)}J(y) = -\frac{\delta}{\delta J(x)}\frac{\delta}{\delta\phi_{cl}(y)}\Gamma$$

$$= -\int\frac{\delta\phi_{cl}}{\delta J}\frac{\delta^2\Gamma}{\delta\phi_{cl}\delta\phi_{cl}} = \int\frac{\delta^2 W}{\delta J\delta J}\frac{\delta^2\Gamma}{\delta\phi_{cl}\delta\phi_{cl}}$$

This means that, in a functional sense,

$$\frac{\delta^2\Gamma}{\delta\phi_{cl}\delta\phi_{cl}} = \left(\frac{\delta^2 W}{\delta J\delta J}\right)^{-1} = D_c^{-1} \quad \text{as expected!}$$

* What about higher order functional derivatives of Γ?

Take a general connected 3-pt function

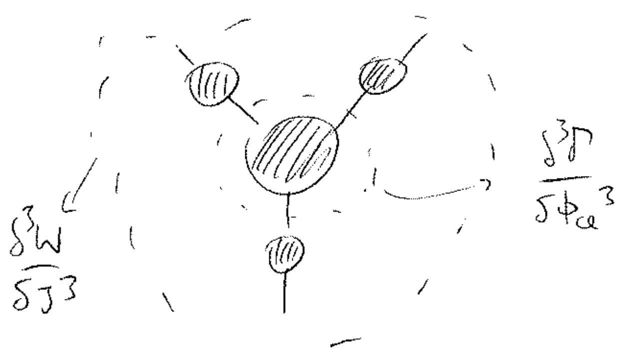
$$\frac{\delta^3(-iW)}{\delta iJ\,\delta iJ\,\delta iJ} = \frac{\delta}{\delta iJ}\left(\frac{\delta^2(-iW)}{\delta iJ\,\delta iJ}\right)$$

$$= \int \frac{\delta \phi_{cl}}{\delta J}\,\frac{\delta}{\delta \phi_{cl}}\left(\frac{\delta^2\Gamma}{\delta\phi_{cl}\,\delta\phi_{cl}}\right)^{-1}$$

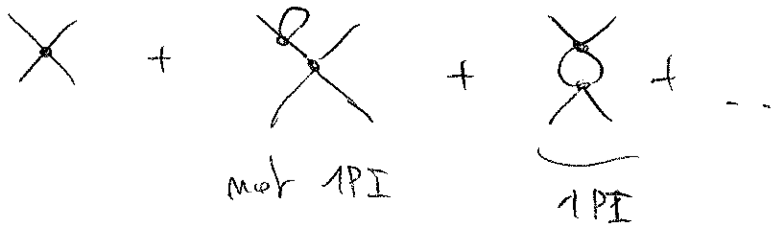
$$= + \int \frac{\delta^2 W}{\delta J\,\delta J} \iint \left(\frac{\delta^2\Gamma}{\delta\phi_{cl}\,\delta\phi_{cl}}\right)^{-1} \cdot \frac{\delta^3\Gamma}{\delta\phi_{cl}\,\delta\phi_{cl}\,\delta\phi_{cl}}\left(\frac{\delta^2\Gamma}{\delta\phi_{cl}\,\delta\phi_{cl}}\right)^{-1}$$

$$= \iiint D\,D\,D\,\frac{\delta^3\Gamma}{\delta\phi_{cl}\,\delta\phi_{cl}\,\delta\phi_{cl}}$$

we thus see that $\frac{\delta^3\Gamma}{\delta\phi_{cl}^3}$ gives the connected 3-pt function stripped of 3 <u>complete</u> propagators, that is not only the free ones, but the resummed ones.



E.g. in ϕ^4 theory, $\mathcal{O}(\lambda^2)$ correction to 4-pt function



- It is thus manifest that Γ is the object that contains the physically most interesting quantities (concerning quantum corrections) of a QFT. All diagrams we will actually compute will refer to Γ, even if implicitly.

(3) Path integral for fermionic fields

We now want to generalize the path integral approach to QFTs with other fields. The generalization to complex scalars, or theories with multiple scalars, is trivial. Let us consider therefore fields with spin higher than zero. The first case is spin $1/2$.
For simplicity, let us stick to a Dirac field (as in QFT I).

$$S_{Dirac} = \int d^4x \left(i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \right)$$

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ Clifford 4×4 matrices

$\to \psi$ 4 components, complex

$\bar{\psi} = \psi^\dagger \gamma^0$

$\gamma^\mu \partial_\mu \equiv \not{\partial}$

Recall that in order to perform canonical quantization in a unitary, causal fashion, with a Hamiltonian bounded from below, we needed to have canonical anticommutation relations between ψ and its conjugate momentum $\psi^\dagger$:

$$\{\psi(\vec{x}), \psi^\dagger(\vec{y})\} = \delta^3(\vec{x}-\vec{y})$$

This leads to creation and annihilation operators that also satisfy anticommutation relations.
It is obviously related to the fermionic statistics, which forbids 2 identical states to superpose
($\{a^\dagger, a^\dagger\} = 0$ implies $(a^\dagger)^2|0\rangle = 0$.)

* In the path integral approach, this translates into considering ψ as an <u>anticommuting field</u> ;
in other words, one can think of ψ as being expanded in an orthonormal basis of functions

$\psi(x) = \sum_i \psi_i \phi_i(x)$ with anticommuting coefficients ψ_i

$\to$ these are called <u>Grassmann numbers</u>

We need to understand calculus with Grassmann (or G-odd) numbers first.

If ξ and χ are 2 Grassmann numbers, then they satisfy $\xi\chi = -\chi\xi$

Since $\xi^2 = \xi\xi = -\xi\xi = -\xi^2 = 0$

Then a function of a G-number has a rather straightforward Taylor expansion:

$f(\xi) = f_0 + f_1\xi$ it truncates as soon as $\xi^2 = 0$.

Derivatives w.r.t. G-variables have to be taken specifying from which side one takes it:

From the left (typically) $\frac{\partial^L}{\partial\xi}(\xi\chi) = \chi$

but of course $\frac{\partial^L}{\partial\xi}(\chi\xi) = -\chi$.

Integrals: we want to define

$\int d\xi\, f(\xi) = \int d\xi\, (f_0 + f_1\xi)$

in such a way that it is invariant under shifts of ξ:

$\int d\xi\, f(\xi+\eta) = \int d\xi\, f(\xi)$

then we see that $\int d\xi\, f_1\eta = 0$

Since we want $\int d\xi\, f(\xi)$ to be linear in $f(\xi)$ we eventually postulate

$\int d\xi\, 1 = 0$, $\int d\xi\, \xi = 1$

→ A Grassmann integral acts much like a derivative.

Note also that $\int d\xi\, \xi = 1$ implies $[d\xi] = [\xi]^{-1}$

* For complex G-variables, we adopt the convention

$$(\xi\chi)^* = \chi^* \xi^* = -\xi^* \chi^*$$

Integrating over the complex G-plane is

$$\int d\xi^* d\xi\ \xi\, \xi^* = 1$$

* We are now ready to perform Gaussian integrals over G-odd variables (note it has to be at least one complex ~~one~~ variable):

$$\int d\xi^* d\xi\ e^{-\alpha \xi^* \xi} = \int d\xi^* d\xi\ (1 + \alpha \xi \xi^*) = \alpha$$

Consider now n such variables ξ_i

$$\int \prod_i d\xi_i^* d\xi_i\ e^{-\xi_i^* K_{ij} \xi_j}$$

going to an orthogonal basis of K_{ij} (Hermitian!) $\xi_i = U_{ij} \xi_j'$

$$\rightarrow \int \prod_i d\xi_i^{*\prime} d\xi_i'\ e^{-\xi_i^{*\prime} k_i \xi_i'} = \int \prod_i \left[d\xi_i^{*\prime} d\xi_i' \left(1 + k_i \xi_i' \xi_i^{*\prime}\right)\right] = \prod_i k_i$$

$$= \det K.$$

To compare with the usual Gaussian integral which, on complex variables, would give

$$\int \prod_i dz_i^* dz_i \, e^{-z_i^* k_{ij} z_j} \propto \frac{1}{\det K}$$

let us also consider some insertions (directly in orthogonal basis to make things simpler):

$$\int \Big[\prod_i d\xi_i^* d\xi_i\Big] \xi_h \xi_\ell^* \, e^{-\sum_i \xi_i^* k_i \xi_i} =$$

$$= \int \Big[\prod_i d\xi_i^* d\xi_i\Big] \xi_h \xi_\ell^* \prod_i (1 + k_i \xi_i \xi_i^*)$$

$$= \int \Big[\prod_{i \neq h,\ell} d\xi_i^* d\xi_i \, k_i \xi_i \xi_i^*\Big) \underbrace{d\xi_h^* d\xi_h \, \xi_h (1 + k_h \cancel{\xi_h \xi_h^*})}_{0} \, \underbrace{d\xi_\ell^* d\xi_\ell \, \xi_\ell^* (1 + k_\ell \cancel{\xi_\ell \xi_\ell^*})}_{0}$$

if $h \neq \ell$

$= 0$

$$= \delta_{h\ell} \int \Big(\prod_{i \neq h} d\xi_i^* d\xi_i \, k_i \xi_i \xi_i^*\Big) d\xi_h^* d\xi_h \, \xi_h \xi_h^* (1 + k_h \cancel{\xi_h \xi_h^*})$$

$$= \delta_{h\ell} \prod_{i \neq h} k_i = \delta_{h\ell} \frac{\det K}{k_h}$$

Thus eventually, in the ratio ~~the~~ of the integral with insertions to the one without, det K cancels and one is left with a result very similar to the usual, G-even/bosonic, one.

* We would like to define now a path integral for fermionic fields, for instance

$$\int \mathcal{D}\bar\psi \mathcal{D}\psi \; e^{i\int d^4x (i\bar\psi \not\partial \psi - m\bar\psi\psi)}$$

The propagator should read

$$\langle 0|T\psi(x)\bar\psi(y)|0\rangle = \frac{\int \mathcal{D}\bar\psi \mathcal{D}\psi \; \psi(x)\bar\psi(y)\, e^{i\int d^4x (i\bar\psi\not\partial\psi - m\bar\psi\psi)}}{\int \mathcal{D}\bar\psi \mathcal{D}\psi \; e^{i\int d^4x (i\bar\psi\not\partial\psi - m\bar\psi\psi)}}$$

In order to evaluate it, we introduce sources also for ψ (and $\bar\psi$): these must be Grassmann (external) fields themselves

$$Z[\eta,\bar\eta] = \int \mathcal{D}\bar\psi \mathcal{D}\psi \; e^{i\int d^4x (i\bar\psi\not\partial\psi - m\bar\psi\psi + \bar\eta\psi + \bar\psi\eta)}$$

Note: $\quad \bar\psi = -\frac{\delta^L}{\delta i\eta}\, i\int d^4x\, \bar\psi\eta \quad , \quad \psi = \frac{\delta^L}{\delta i\bar\eta}\, i\int d^4x\, \bar\eta\psi$

so that

we will drop them

$$\langle \psi(x)\bar\psi(y)\rangle = -\frac{1}{Z[\eta,\bar\eta]} \frac{\delta^L}{\delta i\bar\eta(x)} \frac{\delta^L}{\delta i\eta(y)} Z[\eta,\bar\eta]\Big|_{\eta=0=\bar\eta}$$

Let us compute $Z(\eta,\bar\eta)$.

$$\int d^4x \{\bar\psi(i\not\partial - m)\psi + \bar\eta\psi + \bar\psi\eta\} =$$

$$= \int d^4x \{(\bar\psi + \bar\eta(i\not\partial-m)^{-1})(i\not\partial-m)(\psi + (i\not\partial-m)^{-1}\eta) - \bar\eta(i\not\partial-m)^{-1}\eta\}$$

Let us call $D_F^{(x-y)} \equiv (i\not\partial - m)^{-1}\delta^4(x-y)$ such that

$$(i\not\partial - m)\, D_F^{(x-y)} = \delta^4(x-y) \, .$$

Note that
$$(i\not\partial + m)(i\not\partial - m) = -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2$$
$$= -\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu - m^2 = -\Box - m^2$$

$$(-\Box - m^2)\, D_F = (i\not\partial + m)\,\delta^4(x-y)$$

so that
$$D_F = \frac{i\not\partial + m}{-\Box - m^2}\,\delta^4(x-y) \equiv (i\not\partial + m)\, D$$

↳ Klein-Gordon.

* Performing the integral over the shifted $\psi, \bar\psi$ fields, and canceling it through the normalization, we get

$$Z(\eta,\bar\eta) = e^{-i\int d^4x\, d^4y\, \bar\eta(x) D_F(x-y)\eta(y)}$$

so that

$$\langle \psi(x)\bar\psi(y)\rangle = -\frac{\delta}{\delta i\bar\eta}\frac{\delta}{\delta i\eta}\, e^{-i\int d^4x\, d^4y\, \bar\eta D_F \eta}$$

$$= -i\,\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta\eta}\int d^4x\, d^4y\, \bar\eta D_F \eta$$

$$= +i\,\frac{\delta}{\delta\bar\eta}\left(\bar\eta D_F\right) = i\, D_F(x-y) \qquad \text{as expected.}$$

⚠

Thus up to signs, and D_F instead of D, everything goes through as before.

* For interacting fermions, we can introduce for instance the Yukawa interaction, between a scalar and a Dirac fermion

$$S_{tot} = S_\phi + S_\psi + S_{int}$$

$$= \int d^4x \left\{ \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} m_\phi^2 \phi^2 + i\bar\psi \partial\!\!\!/ \psi - m_\psi \bar\psi\psi + \underbrace{y\,\phi\bar\psi\psi}_{\mathcal{L}_{int}} \right\}$$

For the tree-level 3-pt function we get:

$$\langle \phi(x_1)\psi(x_2)\bar\psi(x_3)\rangle_{int} = \int \mathcal{D}\phi\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\; \phi(x_1)\psi(x_2)\bar\psi(x_3)\, e^{iS_\phi + iS_\psi}\left(1 + i\int d^4x\, y\,\phi\bar\psi\psi + \dots\right)$$

$$= \langle \phi(x_1)\psi(x_2)\bar\psi(x_3)\; i\int d^4x\, y\,\phi(x)\bar\psi(x)\psi(x)\rangle_{free}$$

$$= iy\int d^4x\, \langle\phi(x_1)\phi(x)\rangle\langle\psi(x_2)\bar\psi(x)\rangle\langle\psi(x)\bar\psi(x_3)\rangle$$

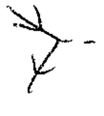
from this factor one reads the relevant Feynman rule.

Usually fermion lines are drawn as solid and oriented

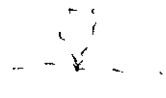
$\langle\psi\bar\psi\rangle \equiv \longrightarrow$

while scalar ones (in this context) are dashed

$\langle\phi\phi\rangle \equiv$ - - - - -

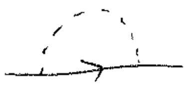
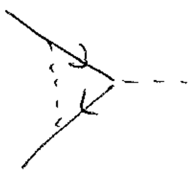
the vertex is thus  iy

* Note that having the scalar interact with a fermion, there are now many more radiative corrections

- to its propagator  (vs )

- to its $\lambda\phi^4$ coupling  (vs )

while both the fermion propagator and Yukawa vertex have also their own radiative corrections

 and. 

(4) Path integral for vector fields

We will be concerned only with the massless case, relevant to electro-magnetism.

$$S_A = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right) \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This action is different from the ones seen before in that it has a redundancy in the off-shell degrees of freedom: we seem to have 4 (A_0, A_1, A_2, A_3) but S_A is perfectly equivalent if we perform the redefinition

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \qquad \text{"gauge transformation"}$$

$$F_{\mu\nu} \to F_{\mu\nu} \quad \text{thus } S_A \to S_A .$$

This gauge freedom can be used to set to zero any one of the components of A_μ.

That S_A did not have all the degrees of freedom of the naive counting had already been seen in QFT I by noting that in canonical quantization there is no momentum conjugate to A_0 : $\frac{\delta S_A}{\delta \dot{A}_0} = 0$.

Yet another way to see that S_A is special is to see the equations of motion:

$$\partial^\mu F_{\mu\nu} = 0 \quad \Leftrightarrow \quad \partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu = 0$$

or $(\Box \delta^\mu_\nu - \partial_\nu \partial^\mu) A_\mu = 0$

The kinetic operator $\Box \delta^\mu_\nu - \partial_\nu \partial^\mu$ has a trivial vanishing eigenfunction: $A_\mu = \partial_\mu \alpha$

$$(\Box \delta^\mu_\nu - \partial_\nu \partial^\mu) \partial_\mu \alpha = \Box \partial_\nu \alpha - \partial_\nu \Box \alpha = 0$$

Thus, since

$$S_A = \int d^4x \left(-\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \partial^\mu A^\nu \right)$$

$$= \int d^4x \left\{ \frac{1}{2} A_\mu (\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu \right\}$$

it implies that

$\int \mathcal{D}A \, e^{iS_A}$ will diverge because of this zero e.v.

* What we would need, is to fix the gauge, or equivalently, add a term to S_A that makes the kinetic operator invertible:

$$S_A \to \int d^4x \left\{ \frac{1}{2} A_\mu (\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right\}$$

$$= \int d^4x \left\{ \frac{1}{2} A_\mu \left(\Box \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right) A_\nu \right\}$$

we would like to find $D_{\mu\nu}$ such that

$$\left[-\Box \eta^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] D_{\nu\rho} = \delta^\mu_\rho \, \delta^4(x-y)$$

in Fourier space

$$\left[+ k^2 \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \right] D_{\nu\rho} = \delta^\mu_\rho$$

Take $D_{\nu\rho} = A(k^2) \eta_{\nu\rho} + B(k^2) k_\nu k_\rho$ by symmetry

$$\underline{+ k^2 A \delta^\mu_\rho} + \cancel{k^2 B k^\mu k_\rho} - \left(1 - \frac{1}{\xi}\right) A k^\mu k_\rho - \left(\cancel{1} - \frac{1}{\xi}\right) B k^2 k^\mu k_\rho = \underline{\delta^\mu_\rho}$$

$$A = \frac{1}{k^2} \qquad B = (\xi - 1) \frac{1}{k^4}$$

$$D_{\mu\nu} = \frac{1}{k^2} \left[\eta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right]$$ ill-defined if $\xi \to \infty$

particularly convenient choices of ξ:

$\xi = 1$ $\quad D_{\mu\nu} = \dfrac{\eta_{\mu\nu}}{k^2}$ Feynman gauge

$\xi = 0$ $\quad D_{\mu\nu} = \dfrac{1}{k^2}\left(\eta_{\mu\nu} - \dfrac{k_\mu k_\nu}{k^2}\right)$ transverse (Landau gauge)

* We are left to show that we can add this term to S_A in the path integral without changing the theory (i.e. the path integral itself).

The following trick is due to Faddeev & Popov (67)

~~Thus~~ We could want to fix the gauge in Z_A by inserting a delta function(al) that imposes some gauge condition $G(A)=0$, such as $\partial^\mu A_\mu=0$.

$$Z_A^{fix} = \int \mathcal{D}A\, \delta(G(A))\, e^{iS_A}$$

Z_A^{fix} would be equivalent to Z_A if we could somehow show that $Z_A = \int \mathcal{D}\alpha \cdot Z_A^{fix}$, i.e. we factor out the trivial (diverging) integral over the gauge redundancy.

This is indeed so: ~~start from~~ recall that

$$1 = \int dx\, \delta(x) = \int dx\, \delta(f(x)) = \int dx\, |f'(x)|\, \delta(f(x))$$

At the functional level

$$1 = \int \mathcal{D}\alpha\, \delta(\alpha) = \int \mathcal{D}\alpha\, \delta(G(A_\alpha)) \left|\det \frac{\delta G(A_\alpha)}{\delta \alpha}\right|$$

$G(A_\alpha)$ is the gauge transformed gauge fixing condition

$$A^\alpha_\mu = A_\mu + \partial_\mu \alpha$$

$$G(A) = \partial_\mu A^\mu \quad \rightarrow \quad G(A^\alpha) = \partial^\mu A_\mu + \Box \alpha$$

So that $\frac{\delta G(A^\alpha)}{\delta \alpha} = \Box$ and $\det \Box$ does not depend on α $\rightarrow$ goes into N

$$\Rightarrow \quad 1 = N \int \mathcal{D}\alpha \, \delta(G(A^\alpha))$$

$$Z_A = N \int \mathcal{D}\alpha \int \mathcal{D}A \, \delta(G(A^\alpha)) \, e^{iS_A}$$

but under $A \rightarrow A^\alpha$ $\mathcal{D}A$ and S_A are invariant thus we can perform the gauge transformation and then relabel A^α to A. We eventually obtain

$$Z_A = N' \int \mathcal{D}A \, \delta(G(A)) \, e^{iS_A}$$ as desired.

* To make contact with the modified action with the ξ term, let us consider a family of gauge fixing conditions

$$G(A) = \partial_\mu A^\mu - \omega(x)$$ (does not change N or N')

and let us integrate, with a Gaussian weight, over them:

$$Z_A = N'' \int \mathcal{D}\omega \, e^{-i\int d^4x \frac{1}{2\xi}\omega^2} \int \mathcal{D}A \, \delta(\partial_\mu A^\mu - \omega) \, e^{iS_A}$$

$$= N'' \int \mathcal{D}A \, e^{iS_A - i\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2}$$ as we postulated before.

* One might be worried that we introduced an arbitrary parameter, ξ, to the theory. However, gauge invariance of the theory must imply that physical quantities (observables) are independent of gauge fixing, and thus of ξ. It is indeed the case, and this can be proven at several levels (of technicality).
Let us here notice that the ξ-dependent part of the photon propagator is

$$D_{\mu\nu} = \dots + \xi \frac{1}{k^4} k_\mu k_\nu \qquad \text{proportional to } k_\mu k_\nu .$$

$\rightarrow$ it is completely longitudinal.

* Now, in order to get $\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle$ let us introduce sources for A_μ:

$$Z[J_\mu] = \int \mathcal{D}A \, e^{iS_A + i\int d^4x \, J^\mu(x) A_\mu(x)}$$

Now notice that because of gauge invariance, J_μ cannot be completely general, but must satisfy:

$$i\int d^4x \, J^\mu A_\mu \rightarrow i\int d^4x \, J^\mu (A_\mu + \partial_\mu \alpha) = i\int d^4x \, (J^\mu A_\mu - \alpha \partial_\mu J^\mu) \overset{!}{=}$$

$$= i\int d^4x \, J^\mu A_\mu \quad \forall \alpha \quad \Leftrightarrow \quad \partial_\mu J^\mu = 0$$

The source for A_μ is a <u>conserved current</u>.

This is consistent with the fact that A_μ, when it couples to other fields, must gauge a symmetry $\to$ it generically couples to conserved currents.

Hence, the longitudinal part of the propagator is not physical and drops in any practical computation.

$\star$ photon 2-pt function: by now standard manipulation.

$$Z[J_\mu] = \int \mathcal{D}A\, e^{i\int \frac{1}{2} A_\mu [\Box\eta^{\mu\nu} - (1-\frac{1}{\xi})\partial^\mu\partial^\nu] A_\nu + J^\mu A_\mu}$$

$$= \int \mathcal{D}A\, e^{i\int \frac{1}{2}(A_\mu - J^\rho D_{\rho\mu})[\Box\eta^{\mu\nu} - (1-\frac{1}{\xi})\partial^\mu\partial^\nu](A_\nu - D_{\nu\sigma}J^\sigma) + i\int \frac{1}{2} J^\mu D_{\mu\nu} J^\nu}$$

$$= N e^{+i\int d^4x\, d^4y\, \frac{1}{2} J^\mu(x) D_{\mu\nu}(x-y) J^\nu(y)}$$

$$\langle A_\mu(x) A_\nu(y)\rangle = \frac{1}{Z} \frac{\delta}{\delta i J^\mu(x)} \frac{\delta}{\delta i J^\nu(y)} Z \Big|_{J=0}$$

$$= -i D_{\mu\nu}(x-y) \qquad \text{as expected}$$

Note that the $-i$ is consistent with the fact that the spatial components of A_μ have a propagator similar to the one of a real (massless) scalar

$$\langle A_1(x) A_1(y)\rangle = \frac{i}{-\Box}\delta^4(x-y) \qquad (\xi = 1 \text{ gauge})$$

* Coupling the vector to "matter" fields: the prime example is obviously QED.

Take the Dirac action:

$$S_F = \int d^4x \left(i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi \right)$$

it is invariant under a global phase rotation under which $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$

α is a constant here.

(This is an abelian $U(1)$ global symmetry.)

If we try to make α local, dependent on x : $\alpha(x)$ we face a problem because

$$\partial_\mu \psi \to \partial_\mu (e^{i\alpha}\psi) = e^{i\alpha}(\partial_\mu\psi + i\,\partial_\mu\alpha \cdot \psi)$$

not homogeneous.

We would like to have a covariant derivative

$D_\mu\psi$ such that $D_\mu\psi \to e^{i\alpha(x)}D_\mu\psi$.

D_μ must contain an extra term than whose transformation compensates the non-homogeneous term:

$$D_\mu = \partial_\mu - iA_\mu \qquad , A_\mu \to A_\mu + \partial_\mu\alpha$$

$$D_\mu\psi = \partial_\mu\psi - iA_\mu\psi \to e^{i\alpha}(\partial_\mu\psi + \cancel{i\partial_\mu\alpha\,\psi} - \cancel{i\partial_\mu\alpha\,\psi} - iA_\mu\psi) = e^{i\alpha}D_\mu\psi$$

A_μ is nothing else than the massless vector above.

The action for QED $(A_\mu + \psi)$ is thus:

$$S_{QED} = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi \right\}$$

$$= S_A + S_F \underbrace{+ e\int d^4x\, A_\mu \bar{\psi}\gamma^\mu\psi}_{S_{int}}$$

e: electric charge $\equiv$ EM coupling.

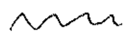
Note that A_μ couples here to $J^\mu_\psi = \bar{\psi}\gamma^\mu\psi e$.

Is it conserved?

$$\partial_\mu J^\mu_\psi = e\bar{\psi}\gamma^\mu\partial_\mu\psi + e\partial_\mu\bar{\psi}\gamma^\mu\psi = 0$$

by $\psi, \bar{\psi}$ eqs. of motion $\quad i\not{\partial}\psi = m\psi$

$$-i\bar{\psi}\overleftarrow{\not{\partial}} = m\bar{\psi}$$

* we denote the propagator of the photon  wavy line

and the vertex is

 $i\gamma^\mu e$

photon self-energy : 

fermion self-energy 

* We can also consider scalar QED.

A complex scalar also has a $U(1)$ global symmetry:

$$\varphi \to e^{i\alpha}\varphi \qquad \varphi^* \to e^{-i\alpha}\varphi$$

$$S_\varphi = \int d^4x \left\{ \partial_\mu\varphi\, \partial^\mu\varphi^* - m^2 \varphi\varphi^* \right\}$$

gauge the symmetry → exactly as before:

opposite sign!

$(\partial_\mu\varphi - ieA_\mu\varphi)(\partial^\mu\varphi^* + ieA^\mu\varphi^*)$

$$S_{\text{Scalar QED}} = \int d^4x \left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\varphi\, D^\mu\varphi^* - m^2\varphi\varphi^* \right\}$$

$$= S_A + S_\varphi + \int d^4x \left[ie A^\mu(\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^*) + e^2 A_\mu A^\mu \varphi\varphi^* \right]$$

we have a linear coupling to ~~the~~ a conserved current

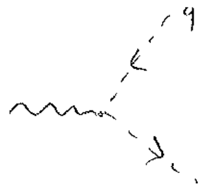
$$J_\varphi^\mu = ie(\varphi^*\partial^\mu\varphi - \varphi\,\partial^\mu\varphi^*)$$

and also a quadratic term of $\mathcal{O}(e^2)$ ensuring the gauge invariance of the $A_\mu J^\mu_\varphi$ term.

* Note that in this QFT, the 3-pt vertex involves also derivatives (to make it Lorentz covariant!)

in momentum space

$$\langle A_\mu(p)\, \varphi(q)\, \varphi^*(q') \rangle \quad \equiv$$



$$ie(q^\mu - q'^\mu)$$

$p = -q - q'$

(5) Symmetries, Ward identities & the path integral

We have just mentioned the importance of conserved currents in QED. Here we will use the path integral to quickly derive the quantum consequences of the presence of global symmetries in a QFT, for instance the quantum equivalent of Noether theorem (the Ward identities) and of the classical eqs. of motion (the Schwinger-Dyson equations).

* Noether theorem: if the action is invariant under a (global) continuous symmetry, there is a conserved current associated to it.

We have just seen two examples: a Dirac fermion and a complex scalar, both invariant under a $U(1)$.

E.g. the fermion

$$S_F = \int d^4x \left\{ i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi \right\} \to \int d^4x \left\{ i\bar{\psi}e^{-i\alpha}\not{\partial}e^{i\alpha}\psi - m\bar{\psi}e^{-i\alpha}e^{i\alpha}\psi \right\}$$

$$= S_F$$

at the infinitesimal level $\psi \to (1+i\alpha)\psi$

$$\delta\psi = i\alpha\psi \qquad \delta\bar{\psi} = -i\alpha\bar{\psi}$$

$$\delta S_F = \int d^4x \left\{ i\delta\bar\psi \not\partial \psi + i\bar\psi \not\partial \delta\psi - m\delta\bar\psi\psi - m\bar\psi\delta\psi \right\}$$

$$= \int d^4x \left\{ \alpha\bar\psi \not\partial\psi - \alpha\bar\psi\not\partial\alpha\psi + im\alpha\bar\psi\psi - im\bar\psi\alpha\psi \right\}$$

$$= 0$$

Let us now make α x-dependent : $\alpha(x)$

since if $\partial_\mu\alpha = 0$ $\delta S_F = 0$, then if $\partial_\mu\alpha \neq 0$

we can only have $\delta S_F \propto \partial_\mu\alpha$

By definition

$$\delta S_F = -\int d^4x\, \partial_\mu\alpha\, J^\mu_\psi$$

integrating by parts $\delta S_F = \int d^4x\, \alpha(x)\, \partial_\mu J^\mu_\psi$

However, $\delta\psi = i\alpha(x)\psi$ can be considered as a general variation of ψ; on the eqs. of motion, S_F should be stationary, for any α : $\delta S_F = 0 \Big|_{EOM}$.

we thus conclude that $\partial_\mu J^\mu_\psi = 0$ on the EOM.

(we already checked that it is indeed the case).

* we can quickly generalize this to any field and any symmetry.

Take a field φ such that under $\delta\varphi = \alpha\,\Delta\varphi$ the action is invariant. Note that this includes the possibility of having some total derivative (when α is constant):

$$\delta\mathcal{L} = \alpha\,\partial_\mu N^\mu$$

When α depends on x, we have

$$\delta S = \int d^4x \left(\frac{\delta\mathcal{L}}{\delta\varphi}\alpha\Delta\varphi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\partial_\mu(\alpha\Delta\varphi)\right)$$

$$= \int d^4x \left(\partial_\mu\alpha \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\Delta\varphi + \alpha\,\partial_\mu N^\mu \right)$$

$$= -\int d^4x\, \alpha\,\partial_\mu\left(-\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\Delta\varphi + N^\mu\right)$$

we have a conserved current $J^\mu = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\Delta\varphi + N^\mu$

$\partial_\mu J^\mu = 0$ on the EOM.

* We now want to know what happens to the conservation equation $\partial_\mu J^\mu = 0$ at the quantum level, i.e. when J^μ is an operator inserted in correlation functions:

$$\langle \partial_\mu J^\mu\, \mathcal{O}_1 \ldots \mathcal{O}_n\rangle = \,?$$

intuitively, we could expect that ~~at~~ any insertion of $\partial_\mu J^\mu$ should render the correlator vanishing.

But recall that $\langle \dots \rangle$ is really $\langle 0 | T(\dots) | 0 \rangle$ and the time ordering might interfere with ∂_μ. The path integral knows all of this...

* Consider $\langle O_1 \dots O_m \rangle = \int \mathcal{D}\varphi \, O_1 \dots O_m \, e^{iS}$

under $\delta\varphi = \alpha \Delta\varphi \rightarrow \Delta O_i = \alpha \Delta O_i$ but any x-dep transformation of the fields can be considered a field redefinition and absorbed by $\mathcal{D}\varphi \to \mathcal{D}\varphi'$, assuming the measure is invariant (there are no anomalies!)

We have then:

$$0 = \int \mathcal{D}\varphi \, \delta \left(O_1 \dots O_m e^{iS} \right)$$

$$= \int \mathcal{D}\varphi \left\{ \delta O_1 \dots O_m + O_1 \delta O_2 \dots O_m + \dots + O_1 \dots \delta O_m + O_1 \dots O_m \, i \delta S \right) e^{iS}$$

$$= \int \mathcal{D}\varphi \left\{ \alpha(x_1) \Delta O_1 \dots O_m + \dots + \alpha(x_m) O_1 \dots \Delta O_m + i \int O_1 \dots O_m \, \alpha(x) \partial_\mu J^\mu \right\} e^{iS}$$

$$= \alpha(x_1) \langle \Delta O_1(x_1) \dots O_m(x_m) \rangle + \alpha(x_m) \langle O_1(x_1) \dots O_m(x_m) \rangle$$
$$+ i \int d^4x \, \alpha(x) \langle \partial_\mu J^\mu(x) O(x_1) \dots O(x_m) \rangle$$

$$= \int d^4x \, \alpha(x) \Big\{ \delta(x - x_1) \langle \Delta O_1(x_1) \dots O_m(x_m) \rangle + \dots + \delta(x - x_m) \langle O_1(x_1) \dots \Delta O_m(x_m) \rangle$$
$$+ i \langle \partial_\mu J^\mu(x) O(x_1) \dots O(x_m) \rangle \Big\}$$

Thus we see that

$$\langle \partial_\mu J^\mu(x)\, O_1(x_1) \ldots O_m(x_m)\rangle = i\,\delta(x-x_1)\,\langle \Delta O(x_1) \ldots O_m(x_m)\rangle + \ldots$$

$$+ i\,\delta(x-x_m)\,\langle O(x_1) \ldots \Delta O_m(x_m)\rangle$$

The expectation that $\partial_\mu J^\mu(x)$ makes a correlator zero is valid as long as x does not hit one of the locations of the other operators → there are contact terms. These are usually called ~~contact term~~ Ward identities, as a generalization of those that appear in QED, i.e.

$$\langle \partial_\mu J^\mu(x)\,\psi(x_1)\,\bar\psi(x_2)\rangle = -ie\,\delta(x-x_1)\,\langle \psi(x_1)\bar\psi(x_2)\rangle$$

$$+ ie\,\delta(x-x_2)\,\langle \psi(x_1)\,\bar\psi(x_2)\rangle$$

(these are usually referred to as Ward-Takahashi)

* Note also that the above general formula gives

$\langle \partial_\mu J^\mu(x)\rangle = 0$ as a 1-pt function

and

$$\langle \partial_\mu J^\mu(x)\, O(y)\rangle = i\,\delta(x-y)\,\langle O(y)\rangle$$

The above relation is non-trivial only if $\langle O\rangle \neq 0$ i.e. if the symmetry is spontaneously broken

* We can also ask ourselves what happens if the equations of motion are inserted in correlation functions. Consider the simplest variation of a field φ:

$\varphi(x) \to \varphi(x) + \varepsilon(x)$ or $\delta\varphi(x) = \varepsilon(x)$

perform such a variation in the path integral definition of an n-pt correlator

$$\langle \varphi(x_1) .. \varphi(x_n) \rangle = \int \mathcal{D}\varphi \, \varphi(x_1) .. \varphi(x_n) \, e^{iS}$$

by the same argument as before, $\mathcal{D}\varphi$ is invariant under such a shift, hence we can write

$$0 = \int \mathcal{D}\varphi \, \delta\left(\varphi(x_1) .. \varphi(x_n) \, e^{iS} \right)$$

$$= \int \mathcal{D}\varphi \Big\{ \delta\varphi(x_1) .. \varphi(x_n) + .. + \varphi(x_1) .. \delta\varphi(x_n) + \\ + \varphi(x_1) .. \varphi(x_n) \, i\delta S \Big\} e^{iS}$$

$$= \int \mathcal{D}\varphi \Big\{ \varepsilon(x_1) .. \varphi(x_n) + .. + \varphi(x_1) .. \varepsilon(x_n) + \\ + i \int d^4x \, \frac{\delta S}{\delta\varphi(x)} \varepsilon(x) \varphi(x_1) .. \varphi(x_n) \Big\} e^{iS}$$

$\Leftrightarrow$

$$\left\langle \frac{\delta S}{\delta\varphi(x)} \varphi(x_1) .. \varphi(x_n) \right\rangle = i\delta(x - x_1) \langle \varphi(x_2) .. \varphi(x_n) \rangle + \\ + .. + i\delta(x - x_n) \langle \varphi(x_1) .. \varphi(x_{n-1}) \rangle$$

Schwinger-Dyson equations

* For our old friend the real scalar

$$\frac{\delta S}{\delta \phi(x)} = (-\Box - m^2)\phi(x)$$

and so we have

0-pt : $\langle (-\Box - m^2)\phi(x) \rangle = 0$

1-pt : $\langle (-\Box - m^2)\phi(x)\,\phi(y) \rangle = i\delta(x-y)$

2-pt : $\langle (-\Box - m^2)\phi(x)\,\phi(x_1)\,\phi(x_2) \rangle = i\delta(x-x_1)\langle \phi(x_2) \rangle$

$\qquad\qquad + i\delta(x-x_2)\langle \phi(x_1) \rangle$

and so on.

Note that for the 1-pt function, our notation really means

$$(-\Box - m^2)\,\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle = i\delta(x-y)$$

so the Schwinger-Dyson eq. in this case is just what defines the propagator of the theory.

This is true in general :

$$\left\langle \frac{\delta S}{\delta \varphi(x)} \varphi(y) \right\rangle = i\delta(x-y)$$

6 Radiative corrections : loops and divergences

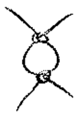
Having established the formalism to compute observables (i.e. correlators) in any QFT, we are now going to perform some computations. As already observed, given an n-pt function (connected, 1PI to make things cleaner) in an interacting QFT, there will be a leading order (i.e. lowest order in the couplings) term and subleading corrections. The latter necessarily include loops, # of internal propagators (we will systematize this rather intuitive statement later on).

E.g. take the $\lambda\phi^4$ theory:

At leading order the 2-pt fn is $O(\lambda^0)$

at order $O(\lambda)$ it is  and so on.

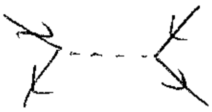
* The 4pt function (1PI) starts at $O(\lambda)$: 

at higher order $O(\lambda^2)$  and so on.

Note that if there are several different couplings one has a double expansion: Yukawa + $\lambda\phi^4$:

both $O(\lambda)$ and $O(y^2)$ arise at one loop.

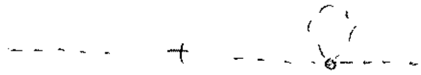
* Some higher pt functions arise at higher order.
  E.g. ψ^4 interaction in Yukawa:

 $O(g^2)$ (⚠ not a term in Γ!)

Terms arising at leading order and with no loops are called tree-level, and can be referred to as representing the semi-classical limit of the QFT amplitudes. In other words, they can be readily evaluated from the Lagrangian:

E.g. the propagators are just the inverse kinetic terms; the vertices  or  just come from the interaction terms; and all other tree-level terms are evaluated by gluing the elements above.

* Radiative corrections are on the other hand genuinely quantum effects that correct the above semi-classical quantities, or generate new terms. This is why we are so interested in computing them, and we will devote such effort to do so ...

* Let us start from the first radiative correction that arises in $\lambda\phi^4$ theory, namely the one on the propagator : 

At the connected 2-pt function level, we have

$$-\frac{i}{4}\lambda\int d^4x\,\langle\phi_1\phi_2\phi_x\phi_x\phi_x\phi_x\rangle = -3i\lambda\int d^4x\,\langle\phi_1\phi_x\rangle\langle\phi_2\phi_x\rangle\langle\phi_x\phi_x\rangle$$

in Fourier space, consider the momentum p entering and exiting is the same, we have then

$$-3i\lambda\,\underbrace{\frac{i}{p^2-m^2}}_{\langle\phi_1\phi_x\rangle}\underbrace{\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}}_{\langle\phi_x\phi_x\rangle}\underbrace{\frac{i}{p^2-m^2}}_{\langle\phi_2\phi_x\rangle}$$

The 1PI part is

$$-iM = -3i\lambda\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}$$

note that it does not depend on p → it is a constant → a shift in m^2 !

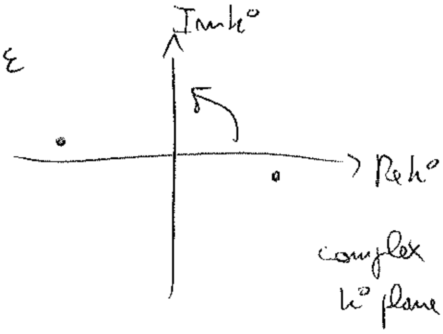
Note also that $\lambda\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}$ has the dimension of a m^2, rightly so. But $[\lambda]=L^0$, then m is the only dimensionful parameter in the theory, apparently. Do we have $M \sim m^2$?

- We have to evaluate $\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2}$

However it has poles where $k^2=m^2$.

Actually, the poles are at $k_0^2 = \vec{k}^2+m^2-i\varepsilon$

thus at $k^0 = \pm\sqrt{\vec{k}^2+m^2} \mp i \frac{\varepsilon}{2\sqrt{\vec{k}^2+m^2}}$



we can rotate the contour to the imaginary axis (counterclockwise), and call the (now imaginary) $k^0 = ik_E^0$

we have then:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2} = -i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2+m^2}$$

$$= -i \int \frac{dk_E \, d^3\Omega}{(2\pi)^4} \frac{k_E^3}{k_E^2+m^2} \qquad \text{spherical coordinates in } \mathbb{R}^4$$

$$= -i \frac{2\pi^2}{(2\pi)^4} \int_0^\infty dk_E \frac{k_E^3}{k_E^2+m^2} \qquad \int d^3\Omega = 2\pi^2 \quad \text{volume of unit } S^3$$

$$= -\frac{i}{(4\pi)^2} \int_0^\infty dk_E^2 \frac{k_E^2}{k_E^2+m^2} \qquad 2k_E dk_E = dk_E^2$$

Now we encounter a seemingly big problem: the integral is obviously divergent! this is embarrassing for a quantum correction, which is supposed to be small.

- In order to get a feel of what is going on, we need to <u>regularize</u> this integral. We will see later a fancier way to do it, for the moment we will content ourselves to introduce a cut-off Λ on the (modulus of the) momentum running in the loop.

We thus evaluate

$$\int_0^{\Lambda^2} du \, \frac{u}{u+m^2} = \int_0^{\Lambda^2} du \left(1 - \frac{m^2}{u+m^2}\right) = \Lambda^2 - m^2 \log \frac{\Lambda^2+m^2}{m^2} \simeq \Lambda^2 \quad \text{if } \Lambda \gg m$$

(Note that in the $m \to 0$ limit, we indeed get Λ^2 exactly).

Putting everything together, we thus get:

$$-iM = 3\lambda \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2} = -\frac{3i\lambda}{(4\pi)^2} \int_0^{\Lambda^2} dk_E^2 \, \frac{k_E^2}{k_E^2+m^2} = -\frac{3i\lambda}{(4\pi)^2} \Lambda^2$$

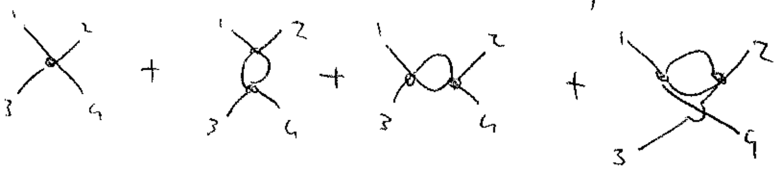
$\Rightarrow$ the correction to m^2 is proportional to:

- the (small) loop factor $\frac{\lambda}{(4\pi)^2}$
- the (large) cut-off scale Λ^2

We thus see that M, which has $[M] = L^{-2}$, is proportional to a scale that is, (rather secretly), also in the definition of the theory, i.e. its UV cut-off.

- We will see how to get rid of this problem through renormalization, however such quadratic dependence on the regulator has some problematic implications.

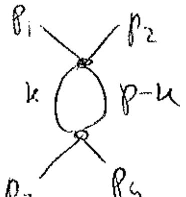
* Let us stick to the $\lambda\phi^4$ theory and let us compute a different radiative correction, to the vertex itself:



(Mandelstam variables $s=(p_1+p_2)^2$ $t=(p_1+p_3)^2$ $u=(p_1+p_4)^2$)
for $2\to2$ scattering

The amputated tree-level diagram was just $-6i\lambda$

The one loop diagrams differ only by the momentum flowing through the loop, they are otherwise identical. We will focus on the first one:



$p_1+p_2+p_3+p_4=0$ (all p_i incoming by convention)

$p=p_1+p_2$ is the momentum flowing through the loop, while k is the internal momentum.

Combinatorial factor: this diagram comes from dropping two iS_{int} in the path integral

Taylor! →

$$\frac{1}{2}\langle\phi_1\phi_2\phi_3\phi_4\left(-\frac{i\lambda}{4}\right)\int\phi_x^4\left(-\frac{i\lambda}{4}\right)\int\phi_y^4\rangle$$

$$=(-i\lambda)^2\frac{1}{2\cdot4\cdot4}\int\langle\phi_1\phi_2\phi_3\phi_4\,\phi_x^4\phi_y^4\rangle$$

$$=(-i\lambda)^2\frac{1}{2\cdot4\cdot4}\int\underset{\substack{\uparrow\\ \text{choice between}\\ x \text{ and } y \text{ vertex}}}{2}\cdot4\langle\phi_1\phi_x\rangle\cdot3\langle\phi_2\phi_x\rangle\cdot4\langle\phi_3\phi_y\rangle\cdot3\langle\phi_4\phi_y\rangle\cdot\underset{\substack{\uparrow\\ \text{2 ways to}\\ \text{combine the}\\ \text{2 remaining}\\ \text{fields at each}\\ \text{vertex.}}}{2}\langle\phi_x\phi_y\rangle^2$$

$$=-18\lambda^2\int\langle\phi_1\phi_x\rangle\langle\phi_2\phi_x\rangle\underbrace{\langle\phi_x\phi_y\rangle^2}_{1PI}\langle\phi_3\phi_y\rangle\langle\phi_4\phi_y\rangle$$

$$-18\lambda^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{(p-k)^2-m^2}$$

we have 2 expressions at the denominator, moreover including $(p-k)^2$ which makes it difficult to use the previous trick of the Wick rotation.

* We thus use a trick by Feynman (as for many tricks..)

$$\int_0^1 dx\,\frac{1}{[ax+b(1-x)]^2}=\int_0^1 dx\,\frac{1}{[(a-b)x+b]^2}=\frac{1}{a-b}\left[-\frac{1}{(a-b)x+b}\right]_0^1$$

$$=\frac{1}{a-b}\left[-\frac{1}{a}+\frac{1}{b}\right]=\frac{1}{a-b}\,\frac{a-b}{ab}=\frac{1}{ab}$$

→ we have a single expression squared at the denominator, at the price of introducing an integration over a (finite range) parameter x.

(Note: the trick can be easily generalized to more expressions at the denominator, by introducing more parameters; expressions like $\frac{1}{a^n b^m}$ can be more easily obtained by deriving the expression above.)

$$18\lambda^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)((p-k)^2-m^2)}=18\lambda^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(p-k)^2x-m^2x+k^2(1-x)-m^2(1-x)]^2}$$

$$=18\lambda^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-2k\cdot p\,x+p^2x-m^2]^2}$$

we still have the $k\cdot p$ term preventing us to integrate over k_E^2.

→ we shift k so that the term linear in k disappears:

$$k = l + xp$$

$$k^2 - 2k\cdot p x + p^2 x = l^2 + \cancel{2l\cdot px} + x^2 p^2 - \cancel{2l\,px} - 2x^2 p^2 + p^2 x$$

$$= l^2 + p^2 x(1-x)$$

the integral becomes $\quad (d^4k = d^4l)$

$$18\lambda^2 \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{1}{[l^2 + p^2 x(1-x) - m^2]^2}$$

we can now take $l^0 = i l_E^0$ and take the same steps as before:

$$\frac{18 i \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^\infty d l_E^2 \frac{l_E^2}{[l_E^2 + m^2 - p^2 x(1-x)]^2}$$

This integral is still divergent, though logarithmically:

$\Delta = m^2 - p^2 x(1-x)$

$$\int_0^{\Lambda^2} du \frac{u}{(u+\Delta)^2} = \int_0^{\Lambda^2} du \left[\frac{1}{u+\Delta} - \frac{\Delta}{(u+\Delta)^2}\right]$$

$$= \left[\log(u+\Delta) + \frac{\Delta}{u+\Delta}\right]_0^{\Lambda^2}$$

$$= \log\frac{\Lambda^2}{\Delta} - 1 \qquad (\Lambda^2 \gg \Delta)$$

Eventually, the radiative correction to $-6i\lambda$ is

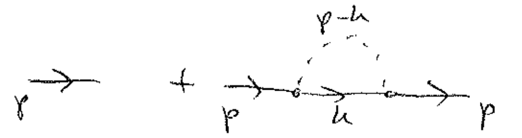
$$-\frac{18 i \lambda^2}{(4\pi)^2} \int_0^1 dx \log\left(\frac{m^2 - s x(1-x)}{\Lambda^2}\right) + (s \leftrightarrow t) + (s \leftrightarrow u)$$

again it diverges, but:

- it corrects λ by $\lambda \cdot \frac{\lambda}{(4\pi)^2}$ (depends on λ itself)
- it depends on the external momenta (s,t,u) but also on m and the cut-off Λ.
- diverges "only" logarithmically (higher degree of divergence would have implied expressions like $\frac{\Lambda^2}{m^2}$ or $\frac{\Lambda^2}{p^2}$ which are not consistent with $m \to 0$, $p^2 \to 0$ limit which should not be ill-defined) - essentially because $[\lambda] = L^0$.

* Our final example of radiative corrections is Yukawa theory: we have ϕ, ψ and $\mathcal{L}_{int} = y\,\phi\bar{\psi}\psi$

we can compute corrections to both ϕ and ψ propagators; let us start from the latter.



Combinatorial factor:

$$(iy)^2 \overbrace{\frac{1}{2}}^{\text{Taylor}} \langle \psi_1 \bar{\psi}_2 \int \phi_x \bar{\psi}_x \psi_x \int \phi_y \bar{\psi}_y \psi_y \rangle$$

$$= (iy)^2 \frac{1}{2} \int (-2) \langle \psi_1 \bar{\psi}_x \rangle \langle \phi_x \phi_y \rangle \langle \psi_x \bar{\psi}_y \rangle (-1) \langle \psi_y \bar{\psi}_2 \rangle$$

$-$ from $\{\bar{\psi}_2, \bar{\psi}_x\} = 0$

2 from choice between x and y vertex

from $\{\psi_y, \bar{\psi}_2\} = 0$

$$= (iy)^2 \int \langle \psi_1 \bar{\psi}_x \rangle \underbrace{\langle \phi_x \phi_y \rangle \langle \psi_x \bar{\psi}_y \rangle}_{1PI} \langle \psi_y \bar{\psi}_2 \rangle$$

1PI diagram is (~~same mass for simplicity~~!)

$$(iy)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m_\phi^2} \frac{i}{\not{k} - m_\psi} = y^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 - m_\phi^2} \frac{\not{k} + m_\psi}{k^2 - m_\psi^2}$$

we perform the same tricks as before; introduce x and take $k = l + px$

$$m^2(x) = m_\phi^2 x + m_\psi^2 (1-x)$$

$$\to y^2 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{\not{l} + \not{p}x + m_\psi}{[l^2 + p^2 x(1-x) - m^2(x)]^2}$$

the integral over $\not{l}$ drops because it is odd in l !

evaluating the integral as before, we get

$$- \frac{iy^2}{(4\pi)^2} \int_0^1 dx\, (\not{p}x + m_\psi) \log \frac{m^2(x) - p^2 x(1-x)}{\Lambda^2}$$

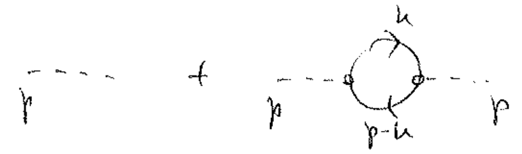
log-divergent correction to $\not{p} - m_\psi$ (recall resummation)

Note that: it is proportional to $\not{p}$ and m_ψ, i.e. there is no linearly divergent term in Λ.

It depends on p (unlike ϕ in $\lambda\phi^4$ theory).

→ prevented by symmetry!

* Finally, let us compute the further correction to the $\langle\phi\phi\rangle$ propagator due to a fermion loop.



$$(iy)^2\frac{1}{2}\langle \phi_1\phi_2\int\phi_x\bar\psi_x\psi_x\int\phi_y\bar\psi_y\psi_y\rangle$$

$$=(iy)^2\int\langle\phi_1\phi_x\rangle\Big[-\langle\psi_x\bar\psi_y\rangle\langle\psi_y\bar\psi_x\rangle\Big]\langle\phi_y\phi_2\rangle$$

from passing $\bar\psi_x$ through $\psi_x, \bar\psi_y, \psi_y$.

1PI is:

$$-(iy)^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\,\frac{i}{\not p-\not k-m}\;\frac{i}{-\not k-m} = -y^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\,\frac{\not p-\not k+m}{(p-k)^2-m^2}\cdot\frac{-\not k+m}{k^2-m^2}$$

$\langle\psi_y\bar\psi_x\rangle$ instead of $\langle\psi_x\bar\psi_y\rangle$

$$=-y^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{tr}\,\frac{(\not p(1-x)-\not\ell+m)(-\not\ell-\not p x+m)}{[\ell^2+p^2x(1-x)-m^2]^2}$$

dropping terms odd in ℓ, and since $\not\ell^2=\ell^2$, this is

$$\cancel{\frac{\pm i y^2}{(4\pi)^2}\int_0^1 dx\int_0^{\Lambda^2}d\ell_E^2\,\ell_E^2\,[\ell_E^2-(\not p x-m)(\not p(1-x)+m)]}$$

$$-y^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{tr}\,\frac{\ell^2-p^2x(1-x)+m\cancel{\not p(1-2x)}+m^2}{[\ell^2+p^2x(1-x)-m^2]^2}$$

when $x\to1-x$: $x(1-x)\to x(1-x)$, $1-2x\to-(1-2x)$ and $\mathrm{tr}\,\gamma^\mu=0$

from $\mathrm{tr}\,\mathbb{1}$ in Clifford space.

$$=\frac{4iy^2}{(4\pi)^2}\int_0^1 dx\int_0^{\Lambda^2}d\ell_E^2\,\frac{\ell_E^2+p^2x(1-x)-m^2}{[\ell_E^2-p^2x(1-x)+m^2]^2}$$

p-dependent

leading divergence is $\frac{4iy^2}{(4\pi)^2}\Lambda^2$ opposite sign w.r.t. $-\frac{3i\lambda}{(4\pi)^2}\Lambda^2$!

(7) The physics of renormalization

[Further reading: Peskin & Schroeder ch.7 ; Weinberg ch.10]

We have just computed quantum corrections to propagators and vertices of QFTs. These clearly affect the effective action (i.e. both kinetic and interaction terms), thus it is legitimate to ask, e.g., what is the physical mass of a particle, or what is the physical field that creates it.

* Consider the propagator, for simplicity in the scalar field theory: $\langle 0| T\,\phi(x)\phi(y)|0\rangle$
  We might want to ask what is the general structure of the propagator in the exact theory, i.e. when all quantum corrections (in the presence of interactions, of course) have been taken into account. For instance, $|0\rangle$ is here the vacuum of the full theory, not just the one of the free theory; it is not usually easy to determine it, we just assume that it exists.

* Here we need to go a bit into the details: we will expand the T-product and consider eigenstates of the Hamiltonian (we loose explicit Lorentz invariance for a while)

Note that because of the Lorentz invariance, use a complete set of eigenstates is written as:

$$\mathbb{1} = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_{p\lambda}} |\lambda_p\rangle\langle\lambda_p|$$

$|\lambda_p\rangle$ is a state of momentum p (can be single- or multi-particle)

$\omega_\lambda = \sqrt{\vec{p}^{\,2} + m_\lambda^2}$ and $\frac{d^3\vec{p}}{\omega_\lambda}$ is Lorentz invariant

the states are normalized as $\langle\lambda_p|\lambda'_{p'}\rangle = \delta_{\lambda\lambda'}(2\pi)^3\, 2\omega_p\, \delta^3(\vec{p}-\vec{p}\,')$

(see QFT I Tinyakov; Peskin)

Then

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_\lambda} \langle 0|\phi(x)|\lambda_p\rangle\langle\lambda_p|\phi(y)|0\rangle$$

with respect to the origin $\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}$

so that

$$\begin{aligned}\langle 0|\phi(x)|\lambda_p\rangle &= \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\lambda_p\rangle \\ &= \langle 0|\phi(0)|\lambda_p\rangle e^{-ip\cdot x} \\ &= \langle 0|\phi(0)|\lambda_0\rangle e^{-ip\cdot x}\end{aligned}$$

since ϕ is invariant under boosts

$\hookrightarrow$ only depends on $p^0 = \sqrt{\vec{p}^{\,2}+m_\lambda^2} = \omega_\lambda$

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_\lambda} e^{-ip\cdot(x-y)} \left|\langle 0|\phi(0)|\lambda\rangle\right|^2$$

Recalling how the contour integral over p^0 works in the $i\varepsilon$ prescription, we get:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m_\lambda^2+i\varepsilon} e^{-ip\cdot(x-y)} |\langle 0|\phi(0)|\lambda\rangle|^2$$

(Lorentz again!)

we write this as

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int_0^\infty dm^2 \rho(m^2) \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot(x-y)}}{p^2-m^2+i\varepsilon}$$

with $\rho(m^2) = \sum_\lambda \delta(m^2-m_\lambda^2) |\langle 0|\phi(0)|\lambda\rangle|^2$

is the spectral density

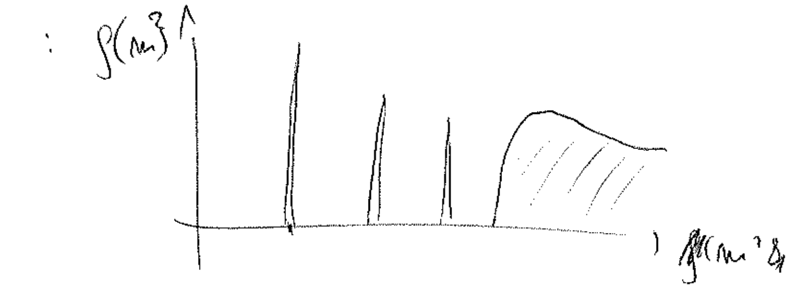
(and this is called the Källén-Lehmann spectral representation)

Note: we could have just postulated such a form for $\langle 0|T\phi(x)\phi(y)|0\rangle$. But in this way we have proven that $\rho(m^2)$ is explicitly <u>real and positive</u>.

* Now the question is what a typical functional form for $\rho(m^2)$. If the state $|\lambda\rangle$ is a single-particle state, then we expect something qualitatively similar to the free propagator: $\rho(m^2) \propto \delta(m^2-m_\lambda^2)$

If $|\lambda\rangle$ is on the other hand a multiparticle state, one expects a continuous function of m^2.

For instance, in the Yukawa example, we have at one-loop  for $p^2 \geq (2m_\psi)^2$ the virtual fermion pair can be on-shell. This appears in $M(p^2)$ as a cut (typically) above the threshold, and a continuous profile for $\rho(m^2)$.

Thus in general : 

Or if $\langle \phi \phi \rangle$ is related to an elementary field, one expects a δ function related to the elementary particle. Before the continuum of multi-particle states, there can be additional δ-fns related to bound states. The presence of the latter is usually related to strong coupling effects.

Note that one can similarly compute correlators of composite operators such as $\langle O(x) O(y) \rangle$ with $O(x) = \bar\psi \psi(x)$. In a strongly coupled theory, such a 2-pt fn can display a $\rho(m^2)$ with a sequence of bound states (composite particles) or resonances (if they have a decay width)

— hadrons are an example.

* In the case of $\langle T\phi(x)\phi(y)\rangle$, let us split the resulting spectral density into the first δ-function and the rest (that, in the simplest, perturbative case can be taken to be the multi-particle continuum).

$$\rho(s) = Z\,\delta(s-m^2) + \bar\rho(s) \qquad \bar\rho(s)=0 \quad s < 4m\, s_0$$
$$s_0 > m^2$$

so that, in Fourier space

$$\langle 0|T\,\phi(p)\phi(-p)|0\rangle = \frac{iZ}{p^2-m^2} + \int_{s_0}^{\infty} ds\,\frac{i\bar\rho(s)}{p^2-s}$$

Note two important things about the first pole: it is to be confronted with the free propagator

$$\langle 0|\phi(p)\phi(-p)|0\rangle_0 = \frac{i}{p^2-m_0^2}$$

here m_0^2 is the mass appearing in the Lagrangian. Above, on the other hand, it is the exact mass of the particle, as it could be eventually measured. The difference between m^2 and m_0^2 is ascribed to quantum corrections, bearing in mind that only m^2 has physical relevance.

Similarly, $Z = |\langle 0|\phi(0)|\lambda\rangle|^2$ with $|\lambda\rangle$ the state corresponding to the physical particle associated to ϕ. ϕ is the field appearing in $\mathcal{L}$, we see it creates the physical particle with a certain normalization.

→ a "physical" field which creates the physical particle with unit normalization needs to be renormalized:

$$|\langle 0|\phi_{ren}|\lambda\rangle|^2 = 1 \qquad \text{if} \qquad \phi = Z^{1/2}\phi_{ren}$$

* We thus see that in making the transition between the original Lagrangian ("bare") and the physical particle propagating, we need to renormalize both the field and the parameters (the mass).

* All of this can be generalized to n-pt correlators (with some technicalities that we will skip). Essentially, this is equivalent to the formalism of the 1PI correlators: we strip the correlators of the external legs, for which we have to take into account the pole at the exact mass, and the normalization of the bare field w.r.t. the renormalized (physical) field:

$$\langle 0|T\phi(x_1)\dots\phi(x_M)|0\rangle = \prod_{i=1}^{M} \frac{i\,Z^{1/2}}{p_i^2-m^2}\; \underbrace{\langle p_1\dots p_M|0\rangle}$$

S-matrix element of one separates k incoming and M-k outgoing physical particles

→ each field is associated to a $Z^{1/2}$ factor.

This is related to the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula.

• E.g. Consider the 4-pt function, and to make things more intuitive split the 4 momenta as 2 incoming and 2 outgoing, as in $2 \to 2$ scattering. Then

$$\langle 0|T\phi(x_1)\ldots\phi(x_4)|0\rangle = \prod_{i=1}^{4} \frac{iZ^{1/2}}{p_i^2-m^2} \langle p_1 p_2 | p_3 p_4\rangle$$

The residue of the (renormalized) poles is the S-matrix related to the 4-pt scattering $\to$ this is the physical value of the 4-pt vertex λ, as opposed to the "bare" value λ_0 as it appears in the Lagrangian. This is another instance of parameter renormalization.

(8) Power counting, divergencies & renormalizability

We have just argued that quantum corrections lead to a renormalization of the fields and the parameters (masses, couplings) of a QFT. But we had already seen that some of these quantum corrections are actually divergent. We will take the point of view that what is finite is the value of the physical, renormalized quantities, and if the corrections diverge then it just means that the "bare" quantities were also diverging to begin with, in a compensating way. In any case, we only measure the physical quantities.

* It is thus important to characterize the divergences of a given QFT. (We will be concerned here only with UV divergences, i.e. those arising for large momenta $|k|\to\infty$; in the presence of massless particles there can be also IR divergences at $|k|\to 0$, but those are dealt with differently, and have a different physical interpretation, related to soft radiation.)

  → it is relevant to know which quantities diverge, how much do they diverge, and how many quantities diverge. For instance, since we have declared that we will reabsorb divergences in the bare parameters of the Lagrangian, there is only a finite number of divergences that can be treated in this way. We thus see that some QFTs will be renormalizable, and some not (those that have a large or infinite number of divergent quantities).

* Recall the example of $\lambda\phi^4$ theory. We have already encountered different types of divergences.

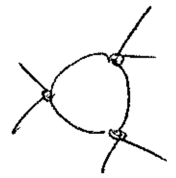
  - 2-pt fn:  $\lambda\int d^4k\,\frac{1}{k^2}\sim\lambda\Lambda^2$ quadratic

    1-loop, 1 internal propagator.

    (we take the limit $|k|\gg|p|, m$, i.e. UV limit)

- 4 pt. fn at 1-loop  $\lambda^2 \int d^4k \, \frac{1}{(k^2)^2} \sim \lambda^2 \log \Lambda^2$

log-divergence.

- 6 pt. fn at 1-loop:  $\lambda^3 \int d^4k \, \frac{1}{(k^2)^3} \sim \lambda^3 \frac{1}{m^2}$

finite!

We see that this theory has, at one-loop, divergences only for the 2-pt and 4-pt functions.

* We want to consider this argument in a more systematic way.

We consider a generalization of $\lambda\phi^4$ theory, in two ways: we take $\lambda\phi^n$ interaction, and consider the QFT in arbitrary dimensions d, instead of focusing in 4:

$$S = \int d^dx \left\{ \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{n} \lambda \phi^n \right\}$$

Most of what we discussed until now carries over trivially to d dimensions. What is more important for the following is to consider the propagator.

The Klein-Gordon eq. in d dimensions is the same

$$(-\square - m^2)\phi = 0 \qquad \text{(free theory)}$$

so that the propagator satisfies

$$(-\square - m^2) D_0(x-y) = \delta^d(x-y)$$

$$\leftrightarrow \quad D(x-y) = \frac{1}{-\square - m^2}\,\delta^d(x-y)$$

or in Fourier space $\qquad D(p) = \dfrac{1}{p^2-m^2}$

i.e. in any d, the propagator (of a scalar) goes like $1/p^2$ at large momenta.

* A generic diagram will be characterized by a set of discrete numbers:

L : the number of loops

N : the number of external lines ("points")

V : the number of vertices ($\lambda\phi^n$)

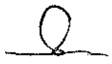
P : the number of internal propagators.

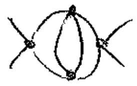
There are relations among these numbers:

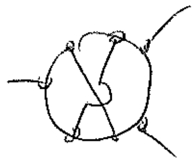
- of topological nature: since each vertex has n lines attached to it, there must be a relation between V and P and N. Since each internal line attaches to two vertices, we have $\quad \underline{nV = 2P + N}$

- dynamical : the number of loops is determined by the other numbers : it equals the number of ~~the~~ d-momentum integrals left after ~~...~~ implementing momentum conservation at every vertex on internal propagators, modulo the overall momentum conservation :

$$L = P - V + 1$$

e.g.  $L = 1 - 1 + 1 = 1$ ✓

 $L = 6 - \cancel{3}4 + 1 = 3$ ✓

less obvious  $L = 9 - 7 + 1 = 3$

* We then define the <u>degree of divergence</u> of a diagram by, essentially, its mass dimension ⇒ this is <u>power counting</u>

  - each loop counts d dimensions because of d^dk, and each internal propagator counts -2 because of $\frac{1}{k^2}$ (large k).
  - we do not take into account for this, of the dimension of coupling !

$$D = dL - 2P$$

we now use the relations above to eliminate L and then P :

$$D = d + (d-2)P - dV$$

$$= d + \frac{d-2}{2}(mV - N) - dV$$

$$D = d - \left(d - m\frac{d-2}{2}\right)V - \frac{d-2}{2}N$$

Now note that in d dimensions, the mass dimension of the field ϕ is determined by its kinetic term to be:

$$[S] = 0 \quad \rightarrow \quad [(\partial\phi)^2] = d \quad \rightarrow 2[\phi] + 2 = d$$

$$\rightarrow [\phi] = \frac{d-2}{2}$$

The interaction vertex has thus dimension $[\phi^m] = m\frac{d-2}{2}$ which implies that $[\lambda] = d - m\frac{d-2}{2}$.

we can thus rewrite the degree of divergence as:

$$\boxed{D = d - [\lambda]V - [\phi]N}$$

it depends on the number of vertices and of external lines.

* let us go back to $d=4$, $m=4$ for a moment. Then $[\phi] = 1$ and $[\lambda] = 0$ so that $D = 4 - N$

→ we reobtain the result that for $N=2$, $D=2$ we have a quadratic divergence, for $N=4$, $D=0$ a logarithmic one, and for $N \geq 6$, $D < 0$ there are no ~~dan~~ more divergences, irrespective of V.

* Sticking to $d=4$, we could take $n=6$. Then $[\lambda] = -2$

and $D = 4 + 2V - N$ and we see that for any N, at high enough order (i.e. large enough V), we will have $D > 0$. Hence any N-pt function at some order will diverge → this is a symptom that it will be difficult to reabsorb all of them in bare quantities.

* Consider now again the general case. Suppose $[\phi] > 0$ ($d > 2$).

Then we have 3 different cases: while increasing N always decreases D, the effect of increasing V crucially depends on ~~M~~ $[\lambda]$:

- if $[\lambda] > 0$, there is only a finite number of diagrams that diverges: those of low V and N. The theory is called <u>super-renormalizable</u>

- if $[\lambda] = 0$, there is a finite number of N-pt functions which have diagrams that diverge, though at any order in V. Such a theory is called <u>renormalizable</u>

- if $[\lambda] < 0$, for any N, there are diagrams with V large enough that diverge. The theory is <u>non-renormalizable</u>

* Note: we speak of <u>superficial</u> degree of divergence, more precisely, because D gives only a rough indication. A diagram can have sub-divergences which make it divergent even if $D < 0$, as e.g.  $D = -2$ but $D' = 2$

or some cancellations (due to symmetries) can make a diagram less divergent or finite even if $D \geq 0$.

(* curious about $d = 2$? In this case $[\phi] = 0$! Then for any n, $[\lambda] = d > 0$ and all such theories are super-renormalizable)

* Examples: $d = 4$ ϕ^4 → renormalizable
ϕ^3 → super renormalizable (but $V < 0$!)
ϕ^6 → non-renormalizable.

Other renormalizable theories are ϕ^n in $d = \frac{2n}{n-2}$

ϕ^3 in $d = 6$, ϕ^6 in $d = 3$

(ϕ^4 in $d = 3$ is super-renormalizable and has $V \geq 0$.)

* this power counting can be straightforwardly generalized to theories including also fermions and vectors, one just has to pay attention to the existence of different internal propagators P (of different dimension: a fermionic one has $D_F \sim \frac{1}{k}$ and so $[D_F] = -1$) and different vertices. But the counting is eventually the same, with the same outcome: renormalizable theories are the ones for which the couplings ~~are~~ have positive or vanishing dimension. Note that the Yukawa coupling and the electric/gauge coupling are both dimensionless.

(9) Counter-terms and renormalization conditions

We now (finally) address renormalized perturbation theory, that is we reformulate the computation of radiative corrections in such a way that physical (renormalized) quantities appear in the results.
We will see that counterterms will manifest themselves in the action, splitting the bare lagrangian from the renormalized one. The splitting is implemented by renormalization conditions. We will also see a more sophisticated regularization of the divergences.

* As usual, we exemplify the procedure with our usual example of $\lambda\phi^4$ theory. Let us start with the usual Lagrangian, that we now declare to be the "bare" one → we put a subscript "b" to all quantities, fields included:

$$\mathcal{L}_{bare} = \frac{1}{2}(\partial_\mu \phi_b)^2 - \frac{1}{2} m_b^2 \phi_b^2 - \frac{1}{4} \lambda_b \phi_b^4$$

But we have seen that the field needs to be renormalized in order to be related to the propagator of the exact quantum particle associated to it:

$$\langle 0 | \phi_b \phi_b | 0 \rangle = \frac{iZ}{p^2 - m_r^2} + \ldots \qquad \phi_b = Z^{\frac{1}{2}} \phi_{ren}$$

so that $\langle 0 | \phi_r \phi_r | 0 \rangle = \frac{i}{p^2 - m_r^2} + \ldots$

we thus rewrite

$$\mathcal{L}_{bare} = \frac{1}{2} Z (\partial_\mu \phi_r)^2 - \frac{1}{2} m_b^2 Z \phi_r^2 - \frac{1}{4} \lambda_b Z^2 \phi_r^4$$

we now split the Lagrangian in the following way:

$$Z = 1 + \delta_Z \quad , \quad m_b^2 Z = m_r^2 + \delta_{m^2} \qquad \lambda_b Z^2 = \lambda_r + \delta_\lambda$$

so that

$$\mathcal{L}_{bare} = \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2} m_r^2 \phi_r^2 - \frac{1}{4} \lambda_r \phi_r^4$$

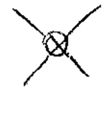
$$+ \frac{1}{2} \delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2} \delta_{m^2} \phi_r^2 - \frac{1}{4} \delta_\lambda \phi_r^4 = \mathcal{L}_{ren} + \mathcal{L}_{c.t.}$$

The first line is the Lagrangian of the theory displaying only the renormalized, physical quantities. The second line contains the difference with respect to the bare Lagrangian, i.e. what we call the <u>counter-terms</u>: the terms that we need to add (subtract, really) to the bare Lagrangian in order to get the renormalized one.

* <u>Renormalized perturbation theory</u> is performed using the counter-terms as perturbations in the computations. Note that there are now "interactions" with only 2 legs.

The quadratic terms $\frac{1}{2}\delta_Z(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_{m^2}\phi_r^2$

lead to an 2-leg vertex $-\otimes-$ $\quad i(p^2\delta_Z - \delta_{m^2})$

while the quartic $-\frac{1}{4}\delta_\lambda\phi_r^4$ leads to  $-i6\delta_\lambda$

→ we denote counterterms in renormalized perturbation theory by a $\otimes$ vertex.

* The coefficients of the counterterms, δ_Z, δ_{m^2} and δ_λ are parameters that need to be adjusted order by order, to cancel the "differences" between bare and renormalized quantities.

such differences are typically divergent, hence the δ's will in general be divergent $\to$ but they are precisely unobservable, since only $\mathcal{L}_{ren}$ is.

Note also that splitting $\mathcal{L}_{bare}$ into $\mathcal{L}_{ren} + \mathcal{L}_{c.t.}$ allows us to introduce only 3 parameters in $\mathcal{L}_{c.t.}$

In other words, we will be able to absorb divergences in at most three classes of diagrams. But we know from power counting that this will be possible for the theory at hand: since $D = 4 - N$, the potentially diverging diagrams have $N \leq 4$ and thus the divergences can be reabsorbed into the coefficients of the quadratic and quartic counterterms.

* There is an arbitrariness in the split "bare $\to$ ren + c.t.". The way to fix such arbitrariness is to set <u>renormalization conditions</u>: in practice enforce that m_r^2, λ_r and ϕ_r are indeed the physical quantities.

However the radiative corrections typically depend on the external momenta $\to$ the renormalization conditions imply a choice of scale in energy.

Let us see an example of such conditions.

- the condition that will determine δ_Z and δ_{m^2} is to require that, at $p^2=m^2$, the exact (connected) propagator is exactly $\frac{i}{p^2-m^2}$. ($m \equiv m_R$ from now on)

Now remember that in general it is $\frac{i}{p^2-m^2-\Pi(p^2)}$

Now expand $\Pi(p^2)$ around $p^2=m^2$:

$$\Pi(p^2) = \Pi(m^2) + \frac{d}{dp^2}\Pi(p^2)\Big|_{p^2=m^2}(p^2-m^2) + \mathcal{O}((p^2-m^2)^2) .$$

we see that $\Pi(m^2)$ would correct m^2, while

$\frac{d}{dp^2}\Pi(p^2)\Big|_{p^2=m^2}$ would correct Z (i.e. the residue)

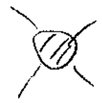
Thus the renormalization conditions are

$$\begin{cases} \Pi(m^2)=0 \\ \frac{d}{dp^2}\Pi(p^2)\Big|_{p^2=m^2} = 0 \end{cases}$$

they will allow us to fix δ_Z and δ_{m^2}

- To determine δ_λ, we need to impose a condition on the exact 4-pt function , at some value of the external momenta.

We consider $1,2 \to 3,4$ scattering and take all of them on shell and collinear; then $s=(p_1+p_2)^2=4m^2$, $t=(p_1+p_3)^2=0$ $u=(p_1+p_4)^2=0$

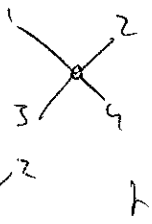
and we ask  $= -6i\lambda$ at such external momenta.

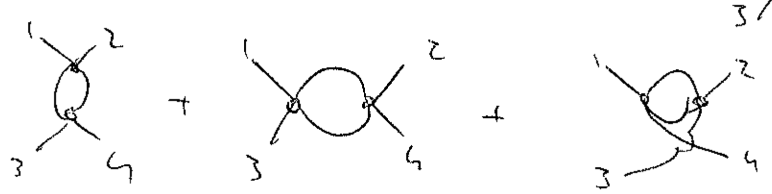
* Note that we could very well take a different point of view and choose a reference scale, let us denote it μ, at which to fix all the conditions, independent on m. It is just a different choice.

* Similarly, in different theories it can be more natural, or easier, to fix the renormalization conditions on other objects that one is computing through radiative corrections.

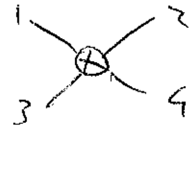
* We now go through a series of examples, actually using the radiative corrections that we had already computed (but implementing an alternative regularization).

The first example is how to fix δ_λ, by computing the one-loop radiative corrections to the 4-pt (1PI) function.

Recall that the corrections to  $-i6\lambda$ were



$V(s)$ + $V(t)$ + $V(u)$

to which we add  $-i6\delta_\lambda$

From the computation in chapter 5, we had

$$V(s) = 18 i \lambda^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{[l_E^2 + m^2 - s x(1-x)]^2}$$

$$= \frac{18 i \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^\infty d l_E^2 \frac{l_E^2}{[l_E^2 + m^2 - s x(1-x)]^2}$$

$$= -\frac{18 i \lambda^2}{(4\pi)^2} \int_0^1 dx \log \frac{m^2 - s x(1-x)}{\Lambda^2}$$ if cut-off at Λ^2 for integral on l_E^2.

The renormalization condition is

$$-i6\lambda + V(s=4m^2) + V(t=0) + V(u=0) - i6\delta_\lambda = -i6\lambda$$

We have thus:

$$6 i \delta_\lambda = V(4m^2) + 2V(0)$$

$$= -\frac{18 i \lambda^2}{(4\pi)^2} \int_0^1 dx \log \frac{m^2}{\Lambda^2}(1-2x)^2 - 2 \cdot \frac{18 i \lambda^2}{(4\pi)^2} \log \frac{m^2}{\Lambda^2}$$

$$\simeq \frac{54 i \lambda^2}{(4\pi)^2} \log \frac{\Lambda^2}{m^2}$$ (we neglected $\int_0^1 dx \log(1-2x)^2 = -2$ over $\log \frac{\Lambda^2}{m^2}$)

$$\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2} \log \frac{\Lambda^2}{m^2}$$ <u>we hid the divergence in δ_λ</u>

* Note that now the 4-pt vertex is finite at this order:

$$V(s) + V(t) + V(u) - 6 i \delta_\lambda = -\frac{18 i \lambda^2}{(4\pi)^2} \int_0^1 dx \log \frac{m^2 - s x(1-x)}{m^2} + (s \leftrightarrow t) + (s \leftrightarrow u)$$

but it is momentum dependent!

Actually, if we want to be extremely precise, we should not neglect any finite term. But any specific prescription of subtracting the infinite part is actually called a <u>renormalization scheme</u>. Observable quantities should be scheme independent.

* We are now going to see an alternative regularization for the same integral. We have seen that the properties of a given QFT change when the dimension of spacetime changes. We are thus going to consider our theory in d dimensions, and analytically continue in d.

Recall

$$V(s) = 18 i \lambda^2 \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{(k_E^2+\Delta)^2} \qquad \Delta = m^2 - sx(1-x)$$

In d dimensions, the radiative correction would be the same but with the following momentum integral:

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2+\Delta)^2}$$

As we are at it, let us be more general and evaluate

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{(k_E^2)^m}{(k_E^2+\Delta)^M} = \frac{V(S^{d-1})}{(2\pi)^d} \int_0^\infty d|k_E| \frac{|k_E|^{d-1} (k_E^2)^m}{(k_E^2+\Delta)^M}$$

$$= \frac{V(S^{d-1})}{2(2\pi)^d} \int_0^\infty dk_E^2 \frac{(k_E^2)^{\frac{d}{2}-1+m}}{(k_E^2+\Delta)^M}$$

Now take $\quad k_E^2 = \Delta x \qquad$ (suppose $\Delta > 0$)

$$\rightarrow \; = \frac{V(S^{d-1})}{2\,(2\pi)^d}\, \Delta^{\frac{d}{2}+m-M} \int_0^\infty dx \, \frac{x^{\frac{d}{2}-1+m}}{(1+x)^M}$$

The power of Δ, the only dimensionful quantity, could have been guessed just by dimensional analysis. The rest are numerical coefficients. $V(S^{d-1})$, the volume of the $d-1$ dimensional unit sphere, is obviously finite. The other factor can contain infinities for certain values of d, m, M. Let us evaluate them in order.

* For $V(S^{d-1})$ we use the d-dimensional Gaussian:

$$\int d^d x_i \, e^{-x_i^2} = \left(\int dx \, e^{-x^2}\right)^d = \pi^{d/2}$$

$$= V(S^{d-1}) \int_0^\infty d\rho \, \rho^{d-1} e^{-\rho^2} \qquad \rho^2 = t$$

$$= \frac{1}{2} V(S^{d-1}) \int_0^\infty dt \, t^{\frac{d}{2}-1} e^{-t}$$

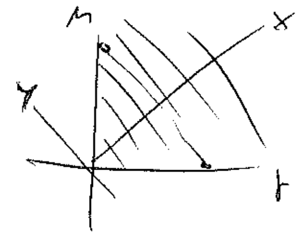
$$= \frac{1}{2} V(S^{d-1}) \, \Gamma\!\left(\frac{d}{2}\right)$$

so that $\displaystyle V(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$ (check with $d=2,3,4$!)

$$\rightarrow \quad \frac{V(S^{d-1})}{2\,(2\pi)^d} = \frac{1}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)}$$

* For the other integral, note that the product of Γ-functions gives:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty dt\, t^{\alpha-1} e^{-t} \int_0^\infty du\, u^{\beta-1} e^{-u}$$



$$= \int_0^\infty dt \int_0^\infty du\, t^{\alpha-1} u^{\beta-1} e^{-(t+u)}$$

$t+u = x \qquad t-u = y \qquad t = \frac{1}{2}(x+y) \qquad u = \frac{1}{2}(x-y)$

$dt\,du = \frac{1}{2} dx\,dy$

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty dx \int_{-x}^{x} dy\, \frac{1}{2^{\alpha+\beta-1}} (x+y)^{\alpha-1}(x-y)^{\beta-1} e^{-x} \qquad y = zx$$

$$= \frac{1}{2^{\alpha+\beta-1}} \int_0^\infty dx\, x^{\alpha+\beta-1} e^{-x} \int_{-1}^{1} dz\, (1+z)^{\alpha-1}(1-z)^{\beta-1} \qquad \begin{matrix} 1+z = 2v \\ 1-z = 2(1-v) \end{matrix}$$

$$= \Gamma(\alpha+\beta) \int_0^1 dv\, v^{\alpha-1}(1-v)^{\beta-1}$$

We define $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \displaystyle\int_0^1 dv\, v^{\alpha-1}(1-v)^{\beta-1}$ Euler B-fn.

* We are left to relate the integral over x to B-function.

take ~~xx~~ $1+x = \frac{1}{t} \qquad x = \frac{1}{t} - 1 = \frac{1-t}{t}$

Then $$\int_0^\infty dx\, \frac{x^{\frac{d}{2}-1+m}}{(1+x)^n} = \int_0^1 dt\, t^{-2} (1-t)^{\frac{d}{2}-1+m}\, t^{-\frac{d}{2}+1-m}\, t^n$$

$$= \int_0^1 dt\, t^{n-m-\frac{d}{2}-1} (1-t)^{m+\frac{d}{2}-1} = B\left(n-m-\frac{d}{2}, m+\frac{d}{2}\right)$$

$$= \frac{\Gamma\left(n-m-\frac{d}{2}\right)\Gamma\left(m+\frac{d}{2}\right)}{\Gamma(n)}$$

* All in all we have the result:

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{(k_E^2)^m}{(k_E^2+\Delta)^n} = \frac{\Delta^{\frac{d}{2}+m-n}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(n-m-\frac{d}{2})\Gamma(m+\frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(n)}$$

The only potential divergences hide in $\Gamma(n-m-\frac{d}{2})$.
Indeed, we expect the integral to be divergent, on dimensional grounds, when $\frac{d}{2}+m-n \geq 0$, i.e. precisely when the argument of the Γ function is zero or negative (and so of integer!).
We regularize the integral taking d to be continuous and close to 4.

* Going back to our integral, we had $m=0$, $n=2$

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(2)} = \frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}} \Gamma(2-\frac{d}{2})$$

Let us actually make the integral exactly dimensionless, as in $d=4$, by multiplying by the appropriate power of an arbitrary scale μ. We the have

$$\mu^{4-d} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2+\Delta)^2} = \frac{1}{(4\pi)^2} \left(\frac{\Delta}{4\pi\mu^2}\right)^{\frac{d}{2}-2} \Gamma(2-\frac{d}{2})$$

Now take $d=4-2\varepsilon$, $\varepsilon \ll 1$.

$$L_1 = \frac{1}{(4\pi)^2}\left(\frac{\Delta}{4\pi\mu^2}\right)^{-\varepsilon}\Gamma(\varepsilon) = \frac{1}{(4\pi)^2}\left(1-\varepsilon\log\frac{\Delta}{4\pi\mu^2}\right)\frac{1}{\varepsilon}\Gamma(1+\varepsilon)$$

$$= \frac{1}{(4\pi)^2}\left(1-\varepsilon\log\frac{\Delta}{4\pi\mu^2}\right)\frac{1}{\varepsilon}(1+\varepsilon\gamma)$$

$\hookrightarrow \Gamma'(1) = 0{,}5772$ Euler-Mascheroni constant

$$\lim_{d\to 4} \mu^{4-d}\int\frac{d^d k_E}{(2\pi)^d}\frac{1}{(k_E^2+\Delta)^2}=\frac{1}{(4\pi)^2}\left(\frac{1}{\varepsilon}+\log 4\pi-\gamma-\log\frac{\Delta}{\mu^2}\right)$$

Compare with the value we obtained with the momentum cut-off:

$$\frac{1}{(4\pi)^2}\left(-\log\frac{\Delta}{\Lambda^2}\right)=\frac{1}{(4\pi)^2}\left(\log\frac{\Lambda^2}{\mu^2}-\log\frac{\Delta}{\mu^2}\right)$$

we have replaced the log-divergence $\log\frac{\Lambda^2}{\mu^2}$ with a single pole in the dimension 4: $\frac{1}{2-\frac{d}{2}}+\underbrace{\log 4\pi-\gamma}$

unobservable: come along...

* Eventually we have:

$$V(p^2)=\frac{18i\lambda^2}{(4\pi)^2}\int_0^1 dx\left(\frac{1}{\varepsilon}+\log 4\pi-\gamma-\log\frac{m^2-p^2x(1-x)}{\mu^2}\right)$$

Then renormalization proceeds exactly as before:

$$6i\delta_\lambda=V(4m^2)+2V(0)$$

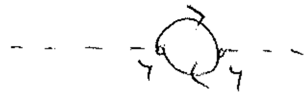
so that the correction to the 4-pt vertex reads

$$\big(V(s)-V(4m^2)\big)+\big(V(t)-V(0)\big)+\big(V(u)-V(0)\big)=$$

$$=-\frac{18i\lambda^2}{(4\pi)^2}\int_0^1 dx\log\frac{m^2-sx(1-x)}{m^2(1-2x)^2}-\frac{18i\lambda^2}{(4\pi)^2}\int_0^1 dx\,\frac{m^2-tx(1-x)}{m^2}-(t\leftrightarrow u)$$

exactly as before (i.e. independent on regularization and its parameters, Λ on one side, ε and μ on the other).

- For field strength and mass renormalization, let us consider the correction to the scalar propagator in Yukawa theory. From the previous computation we had:



$$-i\Pi(p^2) = 4iy^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - \Delta}{(l_E^2+\Delta)^2} \qquad \Delta = m_\varphi^2 - p^2 x(1-x)$$

we use dimensional regularization now:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})\Gamma(1+\frac{d}{2})}{\Gamma(\frac{d}{2})\;\Gamma(2)} \qquad \Gamma(1+\tfrac{d}{2}) = \tfrac{d}{2}\Gamma(\tfrac{d}{2})$$

$$= \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{d/2}} \frac{d}{2} \Gamma(1-\tfrac{d}{2})$$

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{\Delta}{(l_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{d/2}} \Gamma(2-\tfrac{d}{2})$$

Then

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2 - \Delta}{(l_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{d/2}} \left(\frac{d}{2}\Gamma(1-\tfrac{d}{2}) - \Gamma(2-\tfrac{d}{2}) \right) \qquad d = 4-2\varepsilon$$

$$= \frac{\Delta^{1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left[(2-\varepsilon)\Gamma(-1+\varepsilon) - \Gamma(\varepsilon) \right] \qquad \Gamma(-1+\varepsilon) = \frac{\Gamma(\varepsilon)}{-1+\varepsilon}$$

$$= \frac{\Delta}{(4\pi)^2} \left(\frac{\Delta}{4\pi\mu^2} \right)^{-\varepsilon} \left[-\frac{2-\varepsilon}{1-\varepsilon} - 1 \right] \Gamma(\varepsilon)$$

$$= \frac{\Delta}{(4\pi)^2} \left(1 - \varepsilon \log \frac{\Delta}{4\pi\mu^2} \right) (-3-\varepsilon) \left(\frac{1}{\varepsilon} - \gamma \right)$$

$$= -\frac{3\Delta}{(4\pi)^2} \left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - \log \frac{\Delta}{4\pi\mu^2} \right)$$

Note the both the quadratic and log. divergence show up as a simple $\frac{1}{\varepsilon}$ pole. This is a peculiarity of dimensional regularization.

* We now fix the counterterms δ_Z and δ_{m^2} according to the renormalization conditions:

The total correction is

$$-i\Pi(p^2) = -\frac{12 i y^2}{(4\pi)^2}\int_0^1 dx\,(m_\psi^2 - p^2 x(1-x))\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - \log\frac{m_\psi^2 - p^2 x(1-x)}{4\pi\mu^2}\right)$$
$$+ i p^2\delta_Z - i\delta_{m^2}$$

and we require $\Pi(m^2) = 0 \qquad \frac{d}{dp^2}\Pi(p^2)\Big|_{p^2=m^2} = 0$

we have, straightforwardly:

$$-\frac{12 i y^2}{(4\pi)^2}\int_0^1 dx\,(m_\psi^2 - m^2x(1-x))\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - \log\frac{m_\psi^2 - m^2x(1-x)}{4\pi\mu^2}\right) + i m^2\delta_Z - i\delta_{m^2} = 0$$

$$\frac{12 i y^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left(\frac{1}{\varepsilon} - \gamma - \frac{2}{3} - \log\frac{m_\psi^2 - m^2x(1-x)}{4\pi\mu^2}\right) + i\delta_Z = 0$$

then

$$i\delta_Z = -\frac{12 i y^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - 1 - \log\frac{m_\psi^2 - m^2x(1-x)}{4\pi\mu^2}\right)$$

$$i\delta_{m^2} = -\frac{12 i y^2}{(4\pi)^2}\int_0^1 dx\,(m_\psi^2 - m^2x(1-x))\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - \log\frac{m_\psi^2 - m^2x(1-x)}{4\pi\mu^2}\right)$$
$$-\frac{12 i y^2}{(4\pi)^2}\int_0^1 dx\, m^2x(1-x)\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - 1 - \log\frac{m_\psi^2 - m^2x(1-x)}{4\pi\mu^2}\right)$$
$$= -\frac{12 i y^2}{(4\pi)^2}\int_0^1 dx\left\{m_\psi^2\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - \log\frac{m_\psi^2 - m^2x(1-x)}{4\pi\mu^2}\right) - m^2x(1-x)\right\}$$

Eventually :

$$-i\Pi(p^2) = -\frac{12iy^2}{(4\pi)^2}\int_0^1 dx\left(m_\psi^2 - p^2x(1-x)\right)\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - \log\frac{m_\psi^2 - p^2x(1-x)}{4\pi\mu^2}\right)$$

$$-\frac{12iy^2}{(4\pi)^2}\int_0^1 dx\; p^2x(1-x)\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - \log\frac{m_\psi^2 - m^2x(1-x)}{4\pi\mu^2} - 1\right)$$

$$-\frac{12iy^2}{(4\pi)^2}\int_0^1 dx\left\{-m_\psi^2\left(\frac{1}{\varepsilon} - \gamma + \frac{1}{3} - \log\frac{m_\psi^2 - m^2x(1-x)}{4\pi\mu^2}\right) + m^2x(1-x)\right\}$$

$$= +\frac{12iy^2}{(4\pi)^2}\int_0^1 dx\left[\left(m_\psi^2 - p^2x(1-x)\right)\log\frac{m_\psi^2 - p^2x(1-x)}{m_\psi^2 - m^2x(1-x)} + (p^2 - m^2)\,x(1-x)\right]$$

Thus again, we see that δ_Z and δ_{m^2} are divergent, but $\Pi(p^2)$ is eventually finite, though p-dependent.

(10) Renormalization and (gauge) symmetry : QED

We now briefly consider renormalization in a theory which possesses a non-trivial gauge symmetry. We will focus on QED to investigate a few consequences of symmetries (and why we prefer to use dimensional regularization).

We take the bare Lagrangian of QED

$$\mathcal{L}_{bare} = -\frac{1}{4}F_{b\mu\nu}F_b^{\mu\nu} + i\bar{\psi}_b\partial\!\!\!/\psi_b + e_b A_{b\mu}\bar{\psi}_b\gamma^\mu\psi_b - m_b\bar{\psi}_b\psi_b$$

We start by performing fields renormalization on both A_μ and ψ: $\quad A_{b\mu} = Z_A^{1/2} A_\mu \quad , \quad \psi_b = Z_\psi^{1/2} \psi$

(This follows from a completely analogous argument on the exact propagators of vector and spinor fields).

$$\mathcal{L}_{bare} = -\frac{1}{4} Z_A F_{\mu\nu}F^{\mu\nu} + i Z_\psi \bar\psi \not{\partial} \psi + e_b Z_\psi Z_A^{1/2} A_\mu \bar\psi \gamma^\mu \psi - m_b Z_\psi \bar\psi\psi$$

we then split ~~renormal~~ bare quantities into renormalized ones and counterterms:

$$Z_A = 1 + \delta_A \quad , \quad Z_\psi = 1 + \delta_\psi \quad , \quad e_b Z_\psi Z_A^{1/2} = e\underbrace{(1+\delta_e)}_{Z_e} \, , \; m_b Z_\psi = m + \delta_m$$

$$\mathcal{L}_{bare} = \left.-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\bar\psi \not{\partial}\psi + e A_\mu \bar\psi \gamma^\mu \psi - m\bar\psi\psi \quad \right) \mathcal{L}_{ren}$$

$$\left. -\frac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu} + i\delta_\psi \underline{\bar\psi \not{\partial}\psi + e\delta_e A_\mu \bar\psi\gamma^\mu\psi} - \delta_m \bar\psi\psi \quad \right) \mathcal{L}_{c.t.}$$

But here we observe a constraint coming from gauge symmetry: it is obvious that the same gauge symmetry must be at work in $\mathcal{L}_{ren}$ and in $\mathcal{L}_{c.t.}$; hence the covariant derivatives in both pieces of the Lagrangian must coincide. <u>This is only possible if $\delta_\psi = \delta_e$</u>

[It actually follows from the Ward identities (the original ones!).]

This is encoded in the QFT, we do <u>not</u> need to enforce it.

* $Z_\psi = Z_e$ from Ward identities :

recall the Ward identities for $U(1)$ rotations of $\psi, \bar\psi$

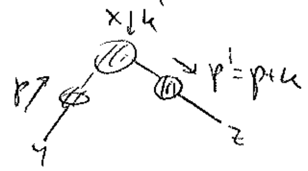
$$\langle \partial_\mu J^\mu(x)\, O_1(x_1) \dots O_n(x_n) \rangle = i\delta(x-x_1)\, \langle \Delta O(x_1) \dots O(x_n) \rangle + \dots$$
$$+ \, i\delta(x-x_n)\, \langle O(x_1) \dots \Delta O(x_n) \rangle$$

$J^\mu(x) = e\bar\psi\gamma^\mu\psi$ at classical level.

Consider now

$$\langle \partial_\mu J^\mu(x)\, \psi(y)\, \bar\psi(z) \rangle = -\delta(x-y)\, \langle \psi(x)\bar\psi(z)\rangle + \delta(x-z) \langle \psi(y)\bar\psi(x)\rangle$$

but we can see $\langle J^\mu(x)\psi(y)\bar\psi(z)\rangle$ as a Green function with two (dressed) legs and a vertex:



the two legs are the propagators appearing on the r.h.s.

we call the vertex $\Gamma^\mu(x)$ to denote γ^μ + quantum corrections

$$\frac{\partial}{\partial x^\mu} \langle \psi(y)\bar\psi(x)\rangle\, \Gamma^\mu(x)\, \langle \psi(x)\bar\psi(z)\rangle = -\delta(x-y)\, \langle\psi(x)\bar\psi(z)\rangle + \delta(x-z) \langle \psi(y)\bar\psi(x)\rangle$$

Go to Fourier space:

$$\langle \psi(y)\bar\psi(x)\rangle \to iD_F(p)$$
$$\langle \psi(x)\bar\psi(z)\rangle \to iD_F(p')$$
$$\Gamma^\mu(x) \to \Gamma^\mu(k)$$

$$-ik_\mu\, iD_F(p)\, \Gamma^\mu(k)\, iD_F(p') = -iD_F(p') + iD_F(p)$$

$$\to \quad k_\mu \Gamma^\mu(k) = D_F(p')^{-1} - D_F(p)^{-1}$$

Now, from $\mathcal{L}_{c.t.}$ we see that among the counterterms we have $i\delta_\psi \not{p}$ for the 2-pt function and $ie\delta_e\gamma^\mu$ for the vertex.

Consider also that the 1PI 2-pt function is given by:

$$\frac{i}{\not{p}-m+\Sigma(p)} = \frac{i}{(\not{p}-m)\left(1+\frac{\Sigma}{\not{p}-m}\right)} = \frac{i}{\not{p}-m}\left(1-\frac{\Sigma}{\not{p}-m}+\dots\right) = \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}\underbrace{(i\Sigma)}_{1PI}\frac{i}{\not{p}-m}+\dots$$

$$D_F(p)^{-1} = \not{p}-m+\Sigma(p)$$

and computing $\Sigma(p)$ at one-loop gives

$i\Sigma = i\Sigma_{\text{finite}} + i\delta_\psi\not{p} + \dots$ so that the divergent piece of Σ, linear in $\not{p}$ is indeed $\delta_\psi\not{p}$.

As for Γ^μ, we have

$$ie\Gamma^\mu = ie\,\Gamma^\mu_{\text{finite}} + ie\,\delta_e\gamma^\mu$$

so that eventually, equating the diverging pieces, we get

$$\not{k}\,\delta_e = \not{p}'\,\delta_\psi - \not{p}\,\delta_\psi \quad \leftrightarrow \quad \delta_e = \delta_\psi \text{ as desired.}$$

* Note that $\delta_e = \delta_\psi$ is tightly related to $\partial_\mu J^\mu = 0$ for $J^\mu = e\bar{\psi}\gamma^\mu\psi$, which in QED is true even in presence of A_μ.

Now $\delta_4 = \delta_e$ implies trivially $Z_4 = Z_e$, so that the relation between the physical (renormalized) charge/coupling and the bare one simplifies:

$$e = \frac{Z_4 Z_A^{1/2}}{Z_e} e_b = Z_A^{1/2} e_b$$

In a way, the renormalization of the coupling is completely encoded in the renormalization of the vector field itself.

* We are thus going to compute the diagram

 at $O(e^2)$

Let us denote this 1PI diagram by $i\Pi_{\mu\nu}(p^\mu)$; it has 2 μ,ν indices because it connects to 2 A_μ, A_ν classical fields (in the effective action formalism).

Actually, it should have the same structure as the kinetic term: $\Pi_{\mu\nu}(p) = (p^2\eta_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$

Note that in this way, $p^\mu \Pi_{\mu\nu}(p) = 0$ it is transverse. This is again a manifestation of the Ward identities, i.e. the conservation of the current at the quantum level.

let us resum the non-1PI diagrams for the photon propagator, and let us choose the Landau gauge ($\xi=0$) for the free propagator: $-iD_{\mu\nu} = \frac{-i}{p^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$

We sum (as matrices) $-iD\sum_{n=0}^{\infty}[i\Pi(-iD)]^n$

But note that $\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$ is a projector:

$$\left(\delta^\mu_\rho - \frac{p^\mu p_\rho}{p^2}\right)\left(\delta^\rho_\nu - \frac{p^\rho p_\nu}{p^2}\right) = \delta^\mu_\nu - 2\frac{p^\mu p_\nu}{p^2} + \frac{p^2 p^\mu p_\nu}{p^4} = \delta^\mu_\nu - \frac{p^\mu p_\nu}{p^2}$$

Thus it factorizes in front of the sum:

$$-i\left(\cancel{\delta}\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}\sum_{n=0}^{\infty}\left(\cancel{p^2}\,\Pi(p^2)\frac{1}{\cancel{p^2}}\right)^n = \frac{-i}{p^2(1-\Pi(p^2))}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

We thus expect $\Pi(p^2)$ to be regular as $p^2\to 0$ (otherwise it would give a mass to the photon!). $\Pi(0)$ will actually give the correction to the residue $\to Z_A$.

* let us now compute it. From the path integral we get:

$$(ie)^2\frac{1}{2}\langle A_{\mu_1}A_{\nu_2}\int A_{\rho x}\bar\psi_x\gamma^\rho\psi_x\int A_{\sigma y}\bar\psi_y\gamma^\sigma\psi_y\rangle$$

$$= -(ie)^2\int\langle A_{\mu_1}A_{\rho x}\rangle\gamma^\rho\langle\psi_x\bar\psi_y\rangle\gamma^\sigma\langle\psi_y\bar\psi_x\rangle\langle A_{\sigma y}A_{\nu 2}\rangle$$

$$i\Pi^{\mu\nu} = -(ie)^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\,\gamma^\mu\frac{i}{\not p-\not k-m}\gamma^\nu\frac{i}{-\not k-m}$$

(cfr.  in Yukawa theory)

$$= -e^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{tr}\,\gamma^\mu(\not p-\not k+m)\gamma^\nu(-\not k+m)}{((p-k)^2-m^2)(k^2-m^2)}$$

(tr: over Clifford space)

* Here also, we again use the Ward identities to show $p_\mu \Pi^{\mu\nu} = 0$:

Note that $\langle \partial_\mu J^\mu(x) J^\nu(y) \rangle = 0$ with no contact terms since $\delta J^\mu = 0$.

Now
$$\langle J^\mu(x) J^\nu(y) \rangle = e^2 \langle \bar{\psi}\gamma^\mu\psi(x)\, \bar{\psi}\gamma^\nu\psi(y) \rangle$$
$$= -e^2 \operatorname{tr} \gamma^\mu \langle \psi(x)\bar{\psi}(y)\rangle \gamma^\nu \langle \psi(y)\bar{\psi}(x)\rangle$$
$$\equiv i\, \Pi^{\mu\nu}_{1\text{-loop}}(x-y)$$

In Fourier space, the Ward identity then reads
$$p_\mu \Pi^{\mu\nu}(p) = 0$$
from which we indeed see that $\Pi^{\mu\nu}$ is required to be transverse.

[This and the previous argument can be found in the book by Srednicki; other more formal proofs based on the effective action Γ can be found in other refs.]

We then introduce the x parameter, shift the internal momentum to $k = l + px$ to get:

$$i\Pi^{\mu\nu} = -e^2\int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{\mathrm{tr}\,\gamma^\mu(\not{l}-\not{p}(1-x)-m)\gamma^\nu(\not{l}+\not{p}x-m)}{[l^2+p^2x(1-x)-m^2]^2}$$

Now in the numerator, drop the terms linear in l, the terms with 3 γ matrices since $\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho = 0$, and consider that $\mathrm{tr}\,\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}$

$$\begin{aligned}\mathrm{tr}\,\gamma^\mu\not{l}\gamma^\nu\not{l} &= -\mathrm{tr}\,\gamma^\mu\gamma^\nu\not{l}\not{l} + 2l^\nu\,\mathrm{tr}\,\gamma^\mu\not{l} \\ &= l^2\, 4\eta^{\mu\nu} + 8l^\mu l^\nu \\ &= 4(2l^\mu l^\nu - l^2\eta^{\mu\nu})\end{aligned}$$

and similarly for $\mathrm{tr}\,\gamma^\mu\not{p}\gamma^\nu\not{p}$

$$i\Pi^{\mu\nu} = -4e^2\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\frac{2l^\mu l^\nu - 2p^\mu p^\nu x(1-x) - [l^2 - p^2x(1-x) - m^2]\eta^{\mu\nu}}{[l^2+px(1-x)-m^2]^2}$$

The last piece of information is how to treat

$$\int\frac{d^4l}{(2\pi)^4}\frac{l^\mu l^\nu}{(l^2-\Delta)^2} = \eta^{\mu\nu} I$$ it has to be proportional to $\eta^{\mu\nu}$ by Lorentz invariance.

then contracting with $\eta_{\mu\nu}$ we find simply

$$I = \frac{1}{4}\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta)^2}$$

Thus finally; also taking $l^0 = il_E^0$

$$i\Pi^{\mu\nu} = -4ie^2\int_0^1 dx\int\frac{d^4l_E}{(2\pi)^4}\frac{\frac{1}{2}l_E^2\eta^{\mu\nu} + [p^2x(1-x)+m^2]\eta^{\mu\nu} - 2p^\mu p^\nu x(1-x)}{[l_E^2+m^2-px(1-x)]^2}$$

* If we introduce a momentum cut-off, we see that we have multiple problems. First of all, the quadratically divergent piece is proportional to $\eta^{\mu\nu}$ only → it is not transverse → it violates the Ward identities.

Furthermore, it would evaluate to

$$i\Pi^{\mu\nu} = -\frac{2ie^2}{(4\pi)^2}\Lambda^2\eta^{\mu\nu} + \ldots$$

which gives a $\Pi(p^2) = -\frac{e^2\Lambda^2}{(2\pi)^2 p^2} + \ldots$

signaling an large (tachyonic!) mass for the photon!

* Let us then use dimensional regularization. We just need to recall here that

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{[\ell^2-\Delta]^2} = \frac{\eta^{\mu\nu}}{d}\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^2}$$

so that

$$i\Pi^{\mu\nu} = -4ie^2\int_0^1 dx\int\frac{d^d\ell_E}{(2\pi)^d}\frac{(1-\frac{2}{d})\ell_E^2\eta^{\mu\nu} + [p^2x(1-x)+m^2]\eta^{\mu\nu} - 2p^\mu p^\nu x(1-x)}{[\ell_E^2+m^2-p^2x(1-x)]^2}$$

we now use the results we obtained before:

$$\int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{d/2}}\Gamma(2-\tfrac{d}{2})$$

$$\int\frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\frac{d}{2}\Gamma(1-\tfrac{d}{2}) = \frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}}\Gamma(2-\tfrac{d}{2})\,\Delta\,\underbrace{\frac{\frac{d}{2}}{1-\frac{d}{2}}}_{= -(1-\frac{2}{d})^{-1}\,!}$$

So that

$$i\Pi^{\mu\nu} = -4ie^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dx\, \Delta^{\frac{d}{2}-2} \Big[-(\cancel{m^2}-p^2x(1-x))\eta^{\mu\nu} + (\cancel{m^2}+p^2x(1-x))\eta^{\mu\nu} - 2p^\mu p^\nu x(1-x) \Big]$$

$$= -\frac{8ie^2}{(4\pi)^2}(p^2\eta^{\mu\nu} - p^\mu p^\nu)\int_0^1 dx\, \Gamma\left(2-\frac{d}{2}\right)\left(\frac{\Delta}{4\pi\mu^2}\right)^{\frac{d}{2}-2} x(1-x)$$

$d = 4-2\varepsilon$

$$\Pi(p^2) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left[\frac{1}{\varepsilon} - \gamma + \log 4\pi - \log\frac{\Delta}{\mu^2}\right]$$

* If we take the renormalization condition to be $\Pi(p^2=0)=0$, then the full correction, including the δ_A counterterm, reads

$$\Pi(p^2) = \frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\log\frac{m^2-p^2x(1-x)}{m^2}$$

* Let us think a bit on this result. To simplify the discussion, consider $-p^2 = Q^2 \gg m^2$, i.e. large spacelike momentum, $\leftrightarrow$ short space separation.

Then $\log\frac{m^2+Q^2x(1-x)}{m^2} \simeq \log\frac{Q^2}{m^2} + \log x(1-x) \simeq \log\frac{Q^2}{m^2}$

and since $\int_0^1 dx\, x(1-x) = \frac{1}{6}$ we have

$$\Pi(Q^2) = \frac{4}{3}\frac{e^2}{(4\pi)^2}\log\frac{Q^2}{m^2}$$

This means that

$$Z_A = \frac{1}{1-\Pi} = \frac{1}{1-\frac{4}{3}\frac{e^2}{(4\pi)^2}\log\frac{Q^2}{m^2}}$$

Recall now that $e = Z_A^{1/2} e_0$

But here, since in this simplified setting $Z_A(Q^2=m^2)=1$ we have that

$$\frac{e^2}{(4\pi)^2} = \frac{\frac{e_0^2}{(4\pi)^2}}{1-\frac{4}{3}\frac{e_0^2}{(4\pi)^2}\log\frac{Q^2}{m^2}}$$

where: e_0 is the value of the coupling when $Q^2=m^2$, and in the denominator we have put e_0^2 which is correct at that order.

→ we see that <u>the coupling e depends on the energy scale</u>!

* It is manifest that the above expression can only be a good approximation as long as $\frac{e^2}{(4\pi)^2}\log\frac{Q^2}{m^2} \ll 1$

But we can compute how much e changes with Q: it is easier to consider

$$\frac{(4\pi)^2}{e^2} = \frac{(4\pi)^2}{e_0^2} - \frac{4}{3}\log\frac{Q^2}{m^2}$$

$$\rightarrow Q^2\frac{d}{dQ^2}\left(\frac{(4\pi)^2}{e^2}\right) = -\frac{4}{3}$$

the inverse coupling decreases (logarithmically) with energy

→ The coupling increases with energy!

Taking the expression literally, we see that we reach an infinite coupling when

$$\log \frac{Q^2}{m^2} = \frac{3}{4} \frac{(4\pi)^2}{e_0^2} = \frac{3}{\cancel{4}} \cdot \frac{\cancel{4}\pi}{\alpha} \approx 3\pi \cdot 137 \sim 1000$$

$$\rightarrow Q \sim e^{500} m_e \quad \text{very large!}$$

This scale, at which a coupling which grows in the UV eventually diverges, is called "Landau pole".

* We will now try to systematize much more the physics of running couplings.

(11) Energy scales and evolution of couplings

We have seen that in renormalized perturbation theory, the radiative corrections are finite, but typically depend on the momenta of the quantity one is computing. It is tempting to interpret this as that the "effective" couplings and fields indeed depend on the energy scale. However we must be a bit more precise, because for instance the 4-pt. 1PI diagram in $\lambda\phi^4$ theory, after renormalization, depends on the 3 invariants s, t and u. What does not depend on the external momenta are the counterterms $\delta_Z, \delta_\lambda \dots$, for the simple reason that they are determined at fixed values of those momenta, through the renormalization conditions.

* We can push this logic a bit further and introduce explicitly a renormalization scale M, which will be taken to be the scale at which we impose the renormalization conditions. Previously we used a physical scale (e.g. the physical mass of the particle), now we use an arbitrary scale so that we can study what happens when we change this scale.

In practice, we will impose the renormalization conditions when all external momentum invariants are fixed by M^2. For practical reasons (as in the QED example before), we will take spacelike momenta. We thus fix, in the $\lambda\phi^4$ theory for instance:

$$\Pi(p^2 = -M^2) = 0 \qquad , \quad \frac{d}{dp^2}\Pi(p^2)\Big|_{p^2=-M^2} = 0$$

and  $= -6i\lambda$ at $s = t = u = -M^2$.

Note that we are a bit forgoing the physical intuition of requiring, for instance, a 2-pt function which has a pole at the physical mass with unit residue (for ϕ_r). Here this will happen at $p^2 = -M^2$, not on-shell. But formally, this is as good as anything else.

* Note also that in any regularization scheme, one introduces a mass scale (Λ in momentum cut-off, μ in dim. reg.). Through the renormalization conditions, one is effectively trading the regularization scale for the renormalization scale M.

* To study the evolution of coupling constants, let us study the n-pt connected Green functions in the bare and the renormalized theory:

$$G^{(n)}_{bare} = \langle 0 | T\, \phi_b(x_1) \ldots \phi_b(x_n) | 0 \rangle$$

$$G^{(n)}_{ren} = \langle 0 | T\, \phi_r(x_1) \ldots \phi_r(x_n) | 0 \rangle$$

- $G^{(n)}_{bare}$ depends on the bare parameters, and on the cut-off of the theory (we consider here a momentum cut-off regularization — the same works for dim. reg.)

$$G^{(n)}_{bare}(x_1, \ldots, x_n; \lambda_b, \Lambda)$$

- After renormalization, we reexpress all the physics in terms of renormalized quantities. It should be independent on the cut-off, but it can depend on the renormalization scale M:

$$G^{(n)}_{ren}(x_1, \ldots, x_n; \lambda_r, M)$$

* Obviously, $G^{(n)}_{bare}$ does not depend on M. But it ~~it~~ is simply related to $G^{(n)}_{ren}$ by field renormalization:

$$\phi_b = Z^{\frac{1}{2}} \phi_r$$

so that

$$G^{(n)}_{bare} = Z^{\frac{n}{2}} G^{(n)}_{ren}$$

Now $G^{(n)}_{ren}$ depends on M both explicitly, and implicitly in λ_r which is a function of λ_b and M. Of course, also Z depends on M because of the renormalization condition of the 2-pt function.

We can then write:

$$M \frac{\partial}{\partial M} G^{(n)}_{bare} = 0 = \frac{n}{2} \frac{M}{Z} \frac{\partial Z}{\partial M} \cdot Z^{\frac{n}{2}} G^{(n)}_{ren} +$$

$$+ Z^{\frac{n}{2}} \left(M \frac{\partial}{\partial M} G^{(n)}_{ren} + M \frac{\partial \lambda}{\partial M} \frac{\partial}{\partial \lambda} G^{(n)}_{ren} \right)$$

We define $\gamma = \frac{1}{2}\frac{M}{Z}\frac{\partial Z}{\partial M} = \frac{1}{2}\frac{\partial \ln Z}{\partial \ln M}$

and $\beta = M\frac{\partial \lambda}{\partial M} = \frac{\partial \lambda}{\partial \ln M}$

we thus have:

$$\left(M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + n\gamma \right) G^{(n)}_{ren} = 0$$

This is called the <u>Callan-Symanzik equation</u>. (70)

* Note that $G^{(n)}_{ren}$ is the connected n-pt function, but not the 1PI. The reason is that the external legs are indeed important when it comes, for instance, to measuring the strength of an interaction. In other words, one cannot avoid measuring the legs. This is why the $n\gamma$ term is important.

* Let us see how this works in a familiar example, $\lambda\phi^4$. Here we will see we can neglect γ at first order, when considering the correction to the vertex: with cut-off Λ, and renormalization conditions at $s=t=u=-M^2$, we get:

$$6i\delta_\lambda = 3V(p^2=-M^2) = -\frac{54i\lambda^2}{(4\pi)^2}\int_0^1 dx \log\frac{m^2+M^2x(1-x)}{\Lambda^2}$$

$$\simeq \frac{54i\lambda^2}{(4\pi)^2}\log\frac{\Lambda^2}{M^2} \qquad (\text{we took } m^2 \ll M^2 \ll \Lambda^2)$$

$$\rightarrow \quad \delta_\lambda = \frac{9\lambda^2}{(4\pi)^2} \log \frac{\Lambda^2}{M^2}$$

This means that, at λ_b fixed, λ changes when M is moved because of the M dependence of δ_λ:

$$\lambda_b = \lambda + \delta\lambda$$

$$M \frac{\partial}{\partial M} \lambda_b = 0 \quad \longrightarrow \quad M \frac{\partial}{\partial M} \lambda = - M \frac{\partial}{\partial M} \delta_\lambda = \frac{18\lambda^2}{(4\pi)^2}$$

This is actually the same statement one obtains through the Callan-Symanzik equation for the 4-pt function:

$$G^{(4)}_{ren} = \left(-6i\lambda + V(s) + V(t) + V(u) - 6i\delta_\lambda \right) (\text{ext. legs})$$

The only explicit dependence on M is in δ_λ; at leading order in λ, only the first term is relevant, the others are all subleading, including the one with γ.

$\rightarrow$ the equation reads

$$\beta \frac{\partial}{\partial \lambda} (-6i\lambda) + M \frac{\partial}{\partial M} (-6i\delta_\lambda) = 0 .$$

$$\beta = - M \frac{\partial}{\partial M} \delta_\lambda = \frac{18\lambda^2}{(4\pi)^2} \qquad \text{it coincides with } M \frac{\partial}{\partial M} \lambda .$$

But as we have seen in QED, it is not always just one counterterm, but also Z factors that contribute to β.

* Reconsider the renormalization of $\lambda\phi^4$ term

we had $\lambda_b \phi_b^4 = \lambda_b Z_\phi^2 \phi_r^4 = \lambda_r \phi_r^4 + \delta_\lambda \phi_r^4$

but let us write it (similarly to QED)

$$\lambda_b Z_\phi^2 = \lambda_r Z_\lambda$$

so that $\lambda_r = \dfrac{Z_\phi^2}{Z_\lambda} \lambda_b$

$$\mu \frac{\partial}{\partial \mu} \lambda_r = \left(2 \frac{\mu}{Z_\phi} \frac{\partial Z_\phi}{\partial \mu} - \frac{\mu}{Z_\lambda} \frac{\partial Z_\lambda}{\partial \mu} \right) \lambda_r$$

$Z_\phi = 1+\delta_\phi$, $Z_\lambda = 1+\delta_\lambda$

$$= \left(2 \mu \frac{\partial}{\partial \mu} \delta_\phi - \mu \frac{\partial}{\partial \mu} \delta_\lambda \right) \lambda_r$$

(where we have redefined $\delta_\lambda \to \lambda_r \delta_\lambda$.)

in pure $\lambda\phi^4$ δ_ϕ is $\mathcal{O}(\lambda^2)$ so it disappears, but it would be there in Yukawa theory, for instance.

* $\gamma = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \delta_\phi$ in Yukawa theory

we had (converting to Λ cut-off)

$$\delta_\phi \simeq + \frac{12 y^2}{(4\pi)^2} \int_0^1 dx \, x(1-x) \log \frac{\mu^2}{\Lambda^2} = + \frac{2y^2}{(4\pi)^2} \log \frac{\mu^2}{\Lambda^2}$$

$$\to \gamma = + \frac{2y^2}{(4\pi)^2}$$

This is called "anomalous dimension" because it gives a correction to the classical dimension of the field:

since $\gamma = \frac{1}{Z}\frac{\partial \ln Z}{\partial \ln M}$ $\quad\rightarrow\quad \ln\frac{Z}{Z_0} = 2\gamma \ln\frac{M}{M_0}$

$$\rightarrow Z = Z_0\left(\frac{M}{M_0}\right)^{2\gamma}$$

and so $\quad \phi_r = Z^{-1/2}\phi_b = Z_0^{-1/2}\left(\frac{M}{M_0}\right)^{-\gamma}\phi_b$

it gives an effective scaling dimension to ϕ_r which is no longer the classical one $[\phi_b] = 1$, but becomes $[\phi_r] = 1-\gamma$.

* In the examples above, we saw that both β and γ are just functions of the couplings, and nothing else. For instance, there is no explicit scale dependence. This can be argued by the fact that they appear in the Callan-Symanzik equations for any Green function, but they are universal: γ is related to the scale dependence of Z, β to the scale dependence of the renormalized coupling. $\rightarrow$ β and γ cannot depend on x_i or p_i, and thus on M either.

* The Callan-Symanzik equation can be trivially generalized to theories with different fields $\rightarrow$ there will be a γ_φ related to each field φ_i

and with different couplings $\to$ there will be a β_i for each coupling λ_i. We only need to pay a little attention to theories where the couplings are dimensionful:

consider $\lambda_n \phi^n$ theory, in d dimensions.

Recall that $[\lambda_n] = d - n\frac{d-2}{2}$ (from power counting)

We define a dimensionless coupling $\bar{\lambda}_n$:

$$\lambda_n = \bar{\lambda}_n M^{d-n\frac{d-2}{2}}$$

Now $\bar{\lambda}_n$ and its β-function can appear in the Callan-Symanzik equation as the other couplings. Note however that there is a tree-level (classical) contribution to the β-function. Indeed, assuming λ_n to be independent on M, we have immediately

$$\beta_{\bar{\lambda}_n} = M \frac{\partial}{\partial M} \bar{\lambda}_n = M \frac{\partial}{\partial M}\left(\lambda_n M^{n\frac{d-2}{2}-d}\right) = \left(n\frac{d-2}{2} - d\right)\bar{\lambda}_n \quad \text{at tree level}$$

* Thus in all generality, we can write for any field φ

$$\gamma_\varphi = \frac{1}{2}\frac{\partial \ln Z_\varphi}{\partial \ln M} = \frac{M}{2}\frac{\partial}{\partial M}\delta_{Z_\varphi}$$

and for a coupling g involving a ~~xxxx~~ number of fields φ $\qquad g_b \prod^{m} \varphi_b = g_b \prod^{m} Z_\varphi^{1/2} \varphi_r = Z_g g_r \prod^{m} \varphi_r$

$$\to \quad g_r = Z_g^{-1} \prod^{m} Z_\varphi^{1/2} g_b = \left(1 - \delta_g + \tfrac{1}{2}\sum \delta_\varphi\right) g_b$$

$$\beta = M \frac{\partial}{\partial M} g_r = g_r M \frac{\partial}{\partial M}\left(-\delta_g + \frac{1}{2}\sum \delta_\varphi\right)$$ at lowest order.

* Example of QED : for the coupling e, we have the vertex with 1 A_μ and 2 ψ :

$$\beta = e M \frac{\partial}{\partial M}\left(-\delta_e + \frac{1}{2}\delta_A + \delta_\psi\right)$$

but recall that $\delta_e = \delta_\psi$ by Ward identities.

so that $\beta = \frac{1}{2} e M \frac{\partial}{\partial M} \delta_A$.

From the QED computation of  we have

$$\delta_A \simeq -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left[\frac{1}{\varepsilon} - \gamma + \log\frac{M^2}{4\pi\mu^2}\right]$$

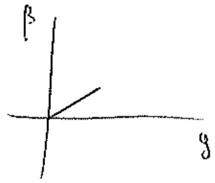
so that $\boxed{\beta_e = \frac{4e^3}{3(4\pi)^2}}$

Note that we have also here $\beta_e > 0$, like for $\lambda\phi^4$ theory.

* Let us explore the different behaviours for β, and what are the consequences for the running of the couplings.

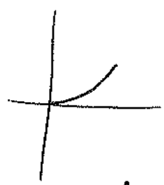
We start by considering, as we have done so far, theories where the coupling is very small.

- if $\beta_g > 0$



$\beta \sim g$
tree level

or



$\beta \sim g^n$
one-loop or higher.

this means that $\mu \frac{\partial}{\partial \mu} g > 0$

g increases with μ

→ there will be a UV scale Λ at which g ceases to be perturbatively small : Landau pole.

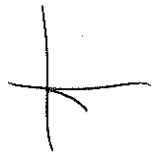
→ on the other hand, going to lower energies (IR) g decreases. If no threshold is met (massive particles), the theory is said to be <u>IR free</u>.

Examples: $\lambda\phi^4$, ~~QED~~ $\beta_\lambda \propto \lambda^2$

QED $\beta_e \propto e^3$

but also $\lambda_M \phi^M$ with $[\lambda_M] < 0$ $\qquad \beta_{\bar\lambda_M} \propto -[\lambda_M]\bar\lambda_M$

- if $\beta_g < 0$

 or 

tree loop

$M\frac{\partial}{\partial M} g < 0$ g <u>decreases</u> when M grows!

Now going to higher energies, g becomes smaller and thus stays perturbative; this is called <u>asymptotic freedom</u>, or <u>UV freedom</u>.

Conversely, at lower energies, at some moment g becomes large $\to$ the scale at which this becomes true is called the <u>dynamical scale</u> Λ (more on this later, about QCD!)

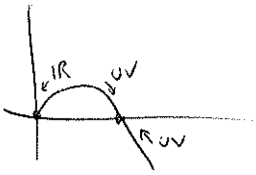
Example: $\lambda_n \phi^n$ with $[\lambda_n] > 0$. But it can happen also at loop level.

* We see that when $\beta > 0$ (resp $\beta < 0$), the evolution of g converges in the IR (resp. in the UV) to zero coupling. This is because $g = 0$ is a <u>fixed pt</u> of the evolution, where $\beta = 0 \Leftrightarrow M\frac{\partial}{\partial M} g = 0$.

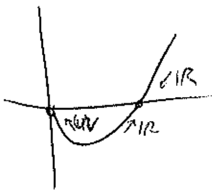
Can there be other fixed points?

We look for other points where $\beta=0$.

we could have $\beta(g)$ of the kind



or



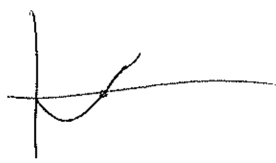
so we could have $\beta=0$ at $g_* \neq 0$: <u>interacting fixed point</u>.

If the evolution between g and g_* takes place for $\beta>0$ then the interacting fixed pt is in the UV, otherwise it is in the IR.

Points where $\beta=0$ are <u>attractors</u> of the evolution (<u>flow</u>) of couplings.

- Example : take $\lambda\phi^4$ in $d=4-2\varepsilon$, now ε small but finite. Then we can sum in β_λ both the tree level and the one-loop contribution (at $d=4$):

$$\beta_\lambda = -2\varepsilon\lambda + \frac{18\lambda^2}{(4\pi)^2}$$



and we find $\beta_\lambda=0$ at $\dfrac{\lambda}{(4\pi)^2} = \dfrac{\varepsilon}{9}$

which, for small enough ε, is well within the perturbative regime. This is the Wilson-Fisher fixed point (72).

* Note that at a fixed point, the theory is rather peculiar: since $\beta = 0$, $\Lambda \frac{\partial}{\partial \Lambda} g = 0$, the coupling is no longer scale dependent. This property can extend to the rest of the theory (γ and so on) so that the theory at the fixed point is scale invariant.

* We would like to end this chapter with a qualitative picture of the evolution of couplings, due to Wilson.

Instead of introducing a renormalization scale, this approach takes the cut-off to be physical.

One defines the path integral as

$$Z = \int \mathcal{D}\phi|_\Lambda \, e^{iS(\phi)}$$

where we explicitly cut-off to only the modes with $|k| \leq \Lambda$ (to properly define this without ambiguities, go to the Euclidean).

Then we can think of slicing further the modes $|k| \leq \Lambda$ into the modes ϕ_{low} with $|k| \leq \Lambda' < \Lambda$ and the modes ϕ_{high} with $\Lambda' < |k| \leq \Lambda$.

Since $S(\phi) = S(\phi_{low}) + S(\phi_{high}, \phi_{low})$ (always possible to split this way)

$$Z = \int \mathcal{D}\phi_{low} \, e^{iS(\phi_{low})} \int \mathcal{D}\phi_{high} \, e^{iS(\phi_{high}, \phi_{low})} \equiv \int \mathcal{D}\phi_{low} \, e^{iS_{eff}(\phi_{low})}$$

→ We have now a theory with a lower cut-off Λ', in which the physics of the high energy modes is encoded in $S_{eff}(\phi_{low})$ → corrections to the couplings of $S(\phi)$ + additional terms (non 1PI).

* One can perform this integrating out of high energy degrees of freedom step by step, and eventually define an evolution of the couplings of S_{eff} which can be shown to be exactly the same as the one given by the Callan-Symanzik equation and β.

* We can thus reconsider the various possibilities $\beta > 0$, $\beta = 0$, $\beta < 0$.

  - when $\beta > 0$ (at tree level, to be definite) then g becomes smaller and smaller as we go down in energies: it is irrelevant in the IR

    Notice that this is exactly the situation for a coupling that we called non-renormalizable!

  - On the contrary, when $\beta < 0$, the coupling grows in the IR and it is relevant there.

- When $\beta = 0$ at tree level, the coupling is called <u>marginal</u>, and if then $\beta > 0$ or $\beta < 0$ at loop level, it is marginally (ir)relevant. (or truly marginal if $\beta = 0$ even at loop level).

* We thus see that from Wilson's perspective in the IR, all QFT's end up being renormalizable, since all non-renormalizable couplings sooner or later become totally irrelevant. This is the point of view of <u>effective field theory</u> (EFT). Of course renormalizability and UV completeness still has a value, from the opposite point of view of the search for a theory of fundamental interactions.

(12) Non-abelian gauge theories

We finally turn to study the class of QFTs which are at the same time the richest and the most relevant to the real world: non abelian gauge theories. Indeed, the Standard Model of particle physics is in great part based on these QFTs. They are a generalization of QED which is completely natural from a group theoretical point of view. But we will see that the non abelian nature of the gauge symmetry will imply non trivial interactions among the gauge bosons. In turn, these will have far-reaching implications at the quantum level, giving us an important example of an class of asymptotically free QFTs. We will also touch upon interesting technicalities inherent to the study of QFT itself.

* In the study of QED, we started with a theory of a Dirac fermion which is symmetric under constant phase rotations $\psi \to e^{i\alpha}\psi$.
This is associated to an abelian $U(1)$ group.
Considering the theory of Lie groups (i.e. continuous groups), a straightforward generalization is to

take a set of Dirac fields ψ_A^A $A = 1..N$
and declare that they belong to a representation of a group G: $\psi_A^A \to U^A{}_B \psi^B$ with U elements of G represented in (unitary) matrix form on the space of ψ.

The Dirac Lagrangian $\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$
is invariant since U are taken to be unitary and
$\bar{\psi} \to \bar{\psi}U^\dagger = \bar{\psi}U^{-1}$ (U and γ^μ commute)

Going to the algebra, we can write $U = e^{i\alpha_a t_a}$
with t_a the hermitian generators of the algebra $\mathcal{A}$; $a = 1..n \equiv \dim G$
α_a are small parameters, hence we can write

$\delta\psi^A = i\alpha_a (t_a)^A{}_B \psi^B$.

The group structure implies $[t_a, t_b] = i f_{abc} t_c$
where f_{abc} are the <u>structure constants</u> and can be taken to be completely antisymmetric.

* Let us now take U to be spacetime dependent.
As with QED, the kinetic term $\bar{\psi}\not{\partial}\psi$ is no longer invariant. We need to replace $\partial_\mu\psi$ with a <u>covariant derivative</u> $D_\mu\psi$ which transforms as
$D_\mu\psi \to U D_\mu\psi$.

we thus introduce a connection A_μ which is in the same representation as U:

$$(D_\mu \psi)^A = \partial_\mu \psi^A - i (A_\mu)^A{}_B \psi^B \,.$$

we have
$$D_\mu \psi \to (\partial_\mu - i A'_\mu) \psi'$$
$$= (\partial_\mu - i A'_\mu) U \psi$$
$$= U \partial_\mu \psi + \partial_\mu U \psi - i A'_\mu U \psi$$

and we require $= U \partial_\mu \psi - i U A_\mu \psi$

Equating the last two lines, and requiring validity for any ψ, implies

$$\partial_\mu U - i A'_\mu U = -i U A_\mu$$

$$\Leftrightarrow \qquad A'_\mu = -i \partial_\mu U \cdot U^{-1} + U A_\mu U^{-1} = i U \partial_\mu U^{-1} + U A_\mu U^{-1}$$

let us now take $U = e^{i\alpha}$, $\alpha = \alpha^a t_a$ in the algebra $\mathcal{A}$

$$\partial_\mu U^{-1} = U^{-1} (-i \partial_\mu \alpha)$$

so that $A'_\mu = \partial_\mu \alpha + (1 + i\alpha) A_\mu (1 - i\alpha) = \partial_\mu \alpha + A_\mu + i[\alpha, A_\mu]$

$\delta A_\mu = \partial_\mu \alpha + i [\alpha, A_\mu]$ $\rightarrow$ A_μ takes values naturally in the algebra $\mathcal{A}$.

If $\mathcal{A}$ is abelian ($U(1)$), then $[\alpha, A_\mu] = 0$ and $\delta A_\mu = \partial_\mu \alpha$ as in QED.

Since $A_\mu \in \mathcal{A}$, we can write it in the t_a basis

$$\delta A^a_\mu t_a = \partial_\mu \alpha_a t_a + i\,[\alpha_a t_a, A_{\mu b} t_b] = \partial_\mu \alpha_a t_a + i \alpha_a A_{\mu b}\, i f_{abc} t_c$$

so that $$\delta A_{\mu a} = \partial_\mu \alpha_a - f_{abc}\alpha_b A_{\mu c} = \partial_\mu \alpha_a + i\alpha_b (T_b)^a{}_c A_{\mu c}$$

where $(T_b)^a{}_c = i f_{abc}$ is the <u>adjoint representation</u>

(i.e. the vector space of the representation of $\mathcal{A}$ is $\mathcal{A}$ itself)

* We have thus generalized the matter part of the QED action to a non abelian symmetry: it still works

$$\mathcal{L} = i\bar\psi \not{D} \psi - m\bar\psi\psi$$

but now $\mathcal{L}_{int} = A_{\mu a}\bar\psi \gamma^\mu t_a \psi$ generalizes $A_\mu \bar\psi \gamma^\mu \psi$.

and, more importantly, the transformation of $A_{\mu a}$ acquires a term linear in A_μ: <u>$\delta A_{\mu a} = \partial_\mu \alpha_a - f_{abc}\alpha_b A_{\mu c}$.</u>

We now would like to write the kinetic term for A_μ. It has to be gauge invariant, but the usual field strength $\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}$ is not because of the linear term in $\delta A_{\mu a}$.

Let us then consider the fact that $[\partial_\mu, \partial_\nu]\psi = 0$

Then $[D_\mu, D_\nu]\psi$ must be simple, and transforms covariantly: $[D_\mu, D_\nu]\psi \to U [D_\mu, D_\nu]\psi$ by the def. of D_μ.

$$[D_\mu, D_\nu]\psi = (\partial_\mu - iA_\mu)(\partial_\nu - iA_\nu)\psi - (\partial_\nu - iA_\nu)(\partial_\mu - iA_\mu)\psi$$

$$= \partial_\mu\partial_\nu\psi - iA_\mu\partial\psi - i\partial_\mu(A_\nu\psi) - A_\mu A_\nu\psi$$
$$- \partial_\mu\partial_\nu\psi + i\partial_\nu(A_\mu\psi) + iA_\nu\partial_\mu\psi + A_\nu A_\mu\psi$$

$$= -i(\partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu A_\nu + iA_\nu A_\mu)\psi$$

$$\equiv -iF_{\mu\nu}\psi$$

we see that indeed $[D_\mu, D_\nu]\psi$ is algebraic on ψ

and we have $\quad -iF'_{\mu\nu}\psi' = -iF'_{\mu\nu}U\psi = -iUF_{\mu\nu}\psi$

so that $\quad F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$ it transforms covariantly.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc} A_{\mu b} A_{\nu c}$$

$$\delta F_{\mu\nu} = i[\alpha, F_{\mu\nu}]$$

$$\delta F_{\mu\nu a} = i\alpha_b (T_b)^a{}_c F_{\mu\nu c}$$

$F_{\mu\nu}$ transforms linearly in the adjoint rep.

Of course if $f_{abc} = 0$ (abelian gauge group) then $F_{\mu\nu}$ is gauge invariant. Here it is only gauge covariant. But it is very easy to build a gauge invariant bilinear with it: $\operatorname{tr} F_{\mu\nu}F^{\mu\nu}$ where the trace is over the adjoint rep., in other words $F_{\mu\nu a}F^{\mu\nu}{}_a$.

* We thus write for the kinetic term of A_μ

$$\mathcal{L}_A = -\frac{1}{4g^2} F_{\mu\nu a} F^{\mu\nu}_a$$

we have written a $\frac{1}{g^2}$ in front because until now we have not included any coupling in the covariant derivatives. We obtain the usual kinetic term and the interactions by rescaling $A_\mu \to g A_\mu$.

Thus the total gauge invariant Lagrangian for A and ψ is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu a} F^{\mu\nu}_a + i \bar{\psi} \not{D} \psi - m \bar{\psi}\psi$$

where
$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g f_{abc} A_{\mu b} A_{\nu c}$$
$$D_\mu \psi = \partial_\mu \psi - i g A_{\mu a} t_a \psi$$

* Note now a very important feature of $\mathcal{L}$:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

with $\mathcal{L}_0$ the quadratic part:

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a})(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$$

and $\mathcal{L}_{int}$ contains terms with only A_μ fields!

$$\mathcal{L}_{int} = -g\, \partial_\mu A_{\nu a} f_{abc} A^\mu_b A^\nu_c - \frac{1}{4} g^2 f_{abc} f_{ade} A_{\mu b} A_{\nu c} A^{\mu d} A^{\nu e}$$
$$+ g A_{\mu a} \bar{\psi} \gamma^\mu t_a \psi$$

→ because of the non-abelian nature, the gauge bosons interact with themselves, unlike the (abelian) photon.

→ we have vertices like  and 
which will lead to radiative corrections of a new kind:

 +  and so on.

→ we expect renormalization and evolution of the couplings also in the theory without matter.

Note finally that gauge symmetry imposes that 3 different vertices ~~has~~ are associated to ~~the~~ one and only coupling g.

(Here for simplicity we assume the group G to be simple. If not, this statement applies to every simple factor.)

(13) Quantization and the need for ghosts

We now proceed to quantize this theory. $\mathcal{L}_{int}$ will straightforwardly provide us with interaction vertices, while $\mathcal{L}_0$ will give us the propagators. However, the vector part of $\mathcal{L}_0$ is exactly equal as the one for QED, replicated n times ($n = \dim G$). Hence it will have the same problem of ~~zero~~ eigenvectors with this eigenvalue. We solved this through gauge fixing, and we saw how to properly do this in the path integral.

* As we did in QED, the gauge is fixed most efficiently by adding $\mathcal{L}_{g.f.} = -\frac{1}{2\xi}(\partial_\mu A^\mu_a)^2$ to $\mathcal{L}_0$.

(Note that this is certainly not the only choice, but a most economical one. Indeed one could imagine a gauge fixing condition with terms of $\mathcal{O}(A^2)$:

$G(A) = \partial_\mu A^\mu + g A_\mu A^\mu = 0$ (schematically)

but then $\mathcal{L}_{g.f.} \propto -\frac{1}{2\xi} G^2$ would introduce new, ξ-dependent interaction terms.)

The free propagator will then be exactly the same as in QED, with an additional δ symbol spanning the adjoint rep:

$$\langle A_{\mu a}(k) A_{\nu b}(-k) \rangle = \frac{-i\delta_{ab}}{k^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

and again $\xi = 1$ (Feynman gauge) will be the most convenient.

* Let us step back and follow the procedure to introduce the gauge-fixing term in the path integral. Recall that we needed to insert in

$Z_A = \int \mathcal{D}A \, e^{iS_A}$

the identity as $1 = \int \mathcal{D}\alpha \, \delta(G(A_\alpha)) \left| \det \frac{\delta G(A_\alpha)}{\delta \alpha} \right|$

where now $\alpha \equiv \alpha_a t_a$ of course.

$$Z_A = \int \mathcal{D}\alpha \mathcal{D}A \, e^{iS_A} \, \delta(G(A_\alpha)) \left| \det \frac{\delta(G(A_\alpha))}{\delta \alpha} \right|$$

However now note that

$$G(A_\alpha) = \partial_\mu (A^\mu + \delta A^\mu) = \partial_\mu A^\mu + \Box \alpha + i \, \partial_\mu [\alpha, A_\mu]$$

it can be rewritten as

$$G(A_\alpha) = \partial_\mu A^\mu + \partial_\mu D^\mu \alpha$$

$, D_\mu \alpha = \partial_\mu \alpha - i [A_\mu, \alpha])$ covariant derivative in the adjoint rep.

Hence $\frac{\delta G(A_\alpha)}{\delta \alpha} = \partial_\mu D^\mu$ but it depends on A_μ!

~~Hen~~ $\to$ it cannot be factored out as an overall factor.

$$Z_A = \int \mathcal{D}\alpha \mathcal{D}A \, e^{iS_A} \, \delta(G(A_\alpha)) \left| \det(\partial_\mu D^\mu(A)) \right|$$

* Then the usual tricks to modify $G(A)$ into $\partial_\mu A^\mu - \omega$ and integrate over ω with a Gaussian weight leads to:

$$Z_A = N \int \mathcal{D}A \, e^{iS_A + iS_{g.f.}} |\det \partial_\mu D^\mu|$$

We still have this A_μ dependence in the det factor, and as it appears now this dependence is non local (i.e. it involves an infinite number of derivatives if one tries to write $\det = e^{\text{tr} \log \det}$).

[Note also that the gauge invariance of the $|\det \partial D^\mu|$ factor is subtle, one needs to take into account the measure on the group – see Serone's notes].

* But we already found a path integral proportional to a det: it was the path integral over fermions. We can thus introduce 2 Grassmann-odd (fermionic) fields such that the path integral over them gives this det:

$$\int \mathcal{D}\bar{c}\mathcal{D}c \, e^{-i\int \bar{c}\partial_\mu D^\mu c} \propto \det \partial D^\mu$$

Note that c and $\bar{c}$ are scalars, and since D^μ is in the adjoint rep, they both need to carry indices of the adjoint rep.

Given that they are scalars but fermionic, they violate the spin-statistics relation, and they better be unphysical, auxiliary fields: they are called <u>ghosts</u> (more precisely, c^a are ghosts and $\bar{c}^a$ anti-ghosts; note that they are not complex conjugates).

We thus have

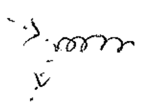
$$Z_A = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c \; e^{iS_A + iS_{g.f.} + iS_{gh}}$$

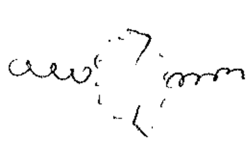
with

$$S_{gh} = -\int d^4x \; \bar{c}_a \partial_\mu (D^\mu c)_a$$

$$= \int d^4x \; \partial_\mu \bar{c}_a \left(\partial^\mu c_a - ig A^\mu_b (T_b)_{ac} c_c\right)$$

$$= \int d^4x \left\{ \partial_\mu \bar{c}_a \partial^\mu c_a + g f_{abc} \partial_\mu \bar{c}_a A^\mu_b c_c \right\}$$

Note that we have introduced a kinetic term for the ghosts, but also an interaction vertex with 2 ghosts and a gauge boson: 

This will allow for virtual ghosts to contribute to radiative corrections!:  ghost loop!

* Including now also the matter fields, the full Lagrangian reads as follows, where we split the quadratic and interacting pieces:

$\hookrightarrow \psi \in \rho$ of G, $\bar\psi \in \rho^*$ of G

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A^\mu_a\right)^2 + i\,\bar\psi_A \partial\!\!\!/ \psi^A + \partial_\mu \bar c_a \partial^\mu c_a$$

$\hookrightarrow$ adj$^* =$ adj

$$\mathcal{L}_{int} = -g\,\partial_\mu A_{\nu a} f_{abc} A^\mu_b A^\nu_c - \frac{1}{4} g^2 f_{abc} f_{ade} A_{\mu b} A_{\nu c} A^\mu_d A^\nu_e$$

$$+ g A_{\mu a} \bar\psi_A \gamma^\mu (t_a)^A{}_B \psi^B \quad + g f_{abc} \cancel{A} \partial_\mu \bar c_a A^\mu_b c_c$$

* In Fourier space, the propagators are:

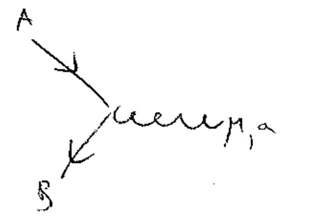
$$\langle A_{\mu a}(k) A_{\nu b}(-k)\rangle = \frac{-i\,\delta_{ab}}{k^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right] \qquad \mu,a \sim\sim\sim\sim \nu,b$$

$$\langle \psi^A(k)\, \bar\psi_B(-k)\rangle = \frac{i\,\delta^A_B}{k\!\!\!/ - m} = i\,\delta^A_B \frac{k\!\!\!/ + m}{k^2 - m^2} \qquad A \longrightarrow B$$

$$\langle c_a(k)\, \bar c_b(-k)\rangle = \frac{i\,\delta_{ab}}{k^2} \qquad a \dashrightarrow b$$

* For the interaction vertices, one has to pay attention to symmetry and the presence of derivatives.

The simplest is the gauge-boson-fermion-fermion coupling:



$$i g \gamma^\mu (t_a)^A{}_B$$

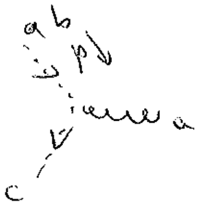
let us work out the gauge boson - ghost-ghost vertex:

$$\langle A_{\mu a} c_b \bar{c}_c \rangle_{int} = \langle A_{\mu a} c_b \bar{c}_c \; i \int g f_{a'b'c'} \partial_\nu \bar{c}_{c'} A^\nu_{a'} c_{b'} \rangle$$

$$= \int i g f_{a'b'c'} \partial_\nu \langle c_b \bar{c}_{c'} \rangle \langle A_{\mu a} A^\nu_{a'} \rangle \langle c_{b'} \bar{c}_c \rangle$$

$$\to i g f_{a'b'c'} \delta_{bc'} (-i)(-p_\nu) \delta_{aa'} \delta^\nu_\mu \delta_{b'c}$$

$$= -g f_{acb} p_\mu = g f_{abc} p_\mu$$



$g f_{abc} p_\mu$ where p_μ is the momentum carried by the incoming ghost with index b.

* Now for the more involved $\partial A A^2$ and A^4 vertices.

$$\langle A_{\mu a} A_{\nu b} A_{\rho c} \rangle_{int} = \langle A_{\mu a} A_{\nu b} A_{\rho c} \; i \int (-g) f_{a'b'c'} \partial_\lambda A_{\sigma a'} A^\lambda_{b'} A^\sigma_{c'} \rangle$$

$$= -i g f_{a'b'c'} \Big\{ \langle A_{\mu a} \partial_\lambda A_{\sigma a'} \rangle \big(\langle A_{\nu b} A^\lambda_{b'} \rangle \langle A_{\rho c} A^\sigma_{c'} \rangle + \langle A_{\nu b} A^\sigma_{c'} \rangle \langle A_{\rho c} A^\lambda_{b'} \rangle \big)$$

$$+ \langle A_{\nu b} \partial_\lambda A_{\sigma a'} \rangle \big(\langle A_{\mu a} A^\lambda_{b'} \rangle \langle A_{\rho c} A^\sigma_{c'} \rangle + \langle A_{\mu a} A^\sigma_{c'} \rangle \langle A_{\rho c} A^\lambda_{b'} \rangle \big)$$

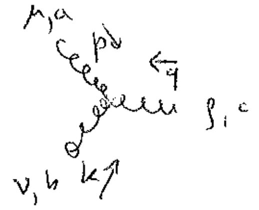
$$+ \langle A_{\rho c} \partial_\lambda A_{\sigma a'} \rangle \big(\langle A_{\mu a} A^\lambda_{b'} \rangle \langle A_{\nu b} A^\sigma_{c'} \rangle + \langle A_{\mu a} A^\sigma_{c'} \rangle \langle A_{\nu b} A^\lambda_{b'} \rangle \big) \Big\}$$

$$\to \quad -i g f_{a'b'c'} \Big\{ \delta_{aa'} \eta_{\mu\sigma} \, i p_\lambda \left(\delta_{bb'} \delta^\lambda_\nu \delta_{cc'} \delta^\sigma_\rho + \delta_{bc'} \delta^\sigma_\nu \delta_{cb'} \delta^\lambda_\rho \right)$$

$$+ \delta_{ba'} \eta_{\nu\sigma} \, i k_\lambda \left(\delta_{ab'} \delta^\lambda_\mu \delta_{cc'} \delta^\sigma_\rho + \delta_{ac'} \delta^\sigma_\mu \delta_{cb'} \delta^\lambda_\rho \right)$$

$$+ \delta_{ca'} \eta_{\rho\sigma} \, i q_\lambda \left(\delta_{ab'} \delta^\lambda_\mu \delta_{bc'} \delta^\sigma_\nu + \delta_{ac'} \delta^\sigma_\mu \delta_{bb'} \delta^\lambda_\nu \right) \Big\}$$

$$= g\,\{ f_{abc}\,\eta_{\mu\rho}\,p_\nu + f_{acb}\,\eta_{\mu\nu}\,p_\rho$$
$$+ f_{bac}\,\eta_{\nu\rho}\,k_\mu + f_{bca}\,\eta_{\mu\nu}\,k_\rho$$
$$+ f_{cab}\,\eta_{\nu\rho}\,q_\mu + f_{cba}\,\eta_{\mu\rho}\,q_\nu \}$$

so that 

$$g\,f_{abc}\{\eta_{\mu\nu}(k-p)_\rho + \eta_{\mu\rho}(p-q)_\nu + \eta_{\nu\rho}(q-k)_\mu\}$$

and finally

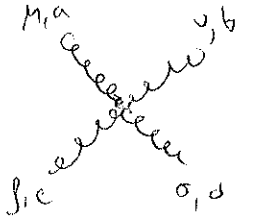
$$\langle A_{\mu a}A_{\nu b}A_{\rho c}A_{\sigma d}\rangle_{int} = \langle A_{\mu a}A_{\nu b}A_{\rho c}A_{\sigma d}\ i\int\left(-\frac{g^2}{4}\right) f_{ea'b'}A_{\alpha a'}A_{\beta b'} f_{ec'd'}A^\alpha_{c'}A^\beta_{d'}\rangle$$

$$= -i\frac{g^2}{4} f_{ea'b'}f_{ec'd'}\int\{2\cdot 2\langle A_{\mu a}A_{\alpha a'}\rangle\}\{\langle A_{\nu b}A_{\beta b'}\rangle\langle A_{\rho c}A^\alpha_{c'}\rangle\langle A_{\sigma d}A^\beta_{d'}\rangle$$
$$+ \langle A_{\nu b}A_{\beta b'}\rangle\langle A_{\rho c}A^\beta_{d'}\rangle\langle A_{\sigma d}A^\alpha_{c'}\rangle + \dots\}$$

$$\to -ig^2 f_{ea'b'}f_{ec'd'}\{\eta_{\mu\alpha}\delta_{aa'}\}\{\eta_{\nu\beta}\delta_{bb'}\delta^\alpha_\rho\delta_{cc'}\delta^\beta_\sigma\delta_{dd'} + \eta_{\nu\beta}\delta_{bb'}\delta^\beta_\rho\delta_{cd'}\delta^\alpha_\sigma\delta_{dc'}$$
$$+ \delta^\alpha_\nu\delta_{bc'}\eta_{\rho\beta}\delta_{cb'}\delta^\beta_\sigma\delta_{dd'} + \delta^\alpha_\nu\delta_{bc'}\delta^\beta_\rho\delta_{cd'}\eta_{\sigma\beta}\delta_{db'}$$
$$+ \delta^\beta_\nu\delta_{bd'}\eta_{\rho\beta}\delta_{cb'}\delta^\alpha_\sigma\delta_{dc'} + \delta^\beta_\nu\delta_{bd'}\delta^\alpha_\rho\delta_{cc'}\eta_{\sigma\beta}\delta_{db'}\}$$

$$= -ig^2\{f_{eab}f_{ecd}\,\eta_{\mu\rho}\eta_{\nu\sigma} + f_{eab}f_{edc}\,\eta_{\mu\sigma}\eta_{\nu\rho} + f_{eac}f_{ebd}\,\eta_{\mu\nu}\eta_{\rho\sigma}$$
$$+ f_{ead}f_{ebc}\,\eta_{\mu\nu}\eta_{\rho\sigma} + f_{eac}f_{edb}\,\eta_{\mu\sigma}\eta_{\nu\rho} + f_{ead}f_{ecb}\,\eta_{\mu\rho}\eta_{\nu\sigma}\}$$

$$= -ig^2\{f_{abe}f_{cde}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) + f_{ace}f_{bde}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$$
$$+ f_{ade}f_{bce}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})\}$$

This is the  vertex.

Now that we have all the propagators and vertices, we can proceed to compute radiative corrections.

(14) Renormalization and the sign of the β function

We start by writing the bare Lagrangian as the renormalized one plus the counterterms. Gauge symmetry will have some implications, though not as strong as in QED.

$$\mathcal{L}_{bare} = -\frac{1}{4}\left(\partial_\mu A^{(b)}_{\nu a} - \partial_\nu A^{(b)}_{\mu a}\right)^2 + i\,\bar{\psi}_{(b)} \not{\partial} \psi_{(b)} + \partial_\mu \bar{c}_{a(b)} \partial^\mu c_{a(b)} - m_{(b)} \bar{\psi}_{(b)} \psi_{(b)}$$

$$- g_{(b)} \partial_\mu A^{(b)}_{\nu a} f_{abc} A^\mu_{b(b)} A^\nu_{c(b)} - \frac{1}{4} g^2_{(b)} f_{abe} f_{cde} A_{\mu a(b)} A_{\nu b(b)} A^\mu_{c(b)} A^\nu_{d(b)}$$

$$+ g_{(b)} A_{\mu a(b)} \bar{\psi}_{(b)} \gamma^\mu t_a \psi_{(b)} + g_{(b)} f_{abc} \partial_\mu \bar{c}_{a(b)} A^\mu_{b(b)} c_{c(b)}$$

we have added a mass term for the fermionic fields, and we have not written the gauge-fixing terms that can be shown not to be renormalized (see Srednicki).

Now write $A_{\mu a(b)} = Z_A^{1/2} A_{\mu a}$, $\psi_{(b)} = Z_\psi^{1/2} \psi$

$c_{a(b)} = Z_c^{1/2} c_a$ $\quad \bar{c}_{a(b)} = Z_c^{1/2} \bar{c}_a$

(we do not write an "r" on the renormalized quantities).

Inspecting the non-kinetic terms, we can write the bare couplings (and mass) in terms of the renormalized ones, introducing a Z factor for each vertex (note that of course, f_{abc} and t_a, as γ_μ, are fixed by group theory and <u>cannot</u> change):

$$m_{(b)} Z_\psi = m Z_m$$

$$g_{(b)} Z_A^{3/2} = g Z_{g(A^3)} \quad , \quad g_{(b)}^2 Z_A^2 = g^2 Z_{g^2(A^4)}$$

$$g_{(b)} Z_A^{1/2} Z_\psi = g Z_{g(A\psi^2)} \qquad g_{(b)} Z_A^{1/2} Z_c = g Z_{g(Ac^2)}$$

$\mathcal{L}_{ren}$ will be exactly as $\mathcal{L}_{bare}$, but where we drop all the (b) indices.

$\mathcal{L}_{c.t.}$ will have a similar structure with $\delta = Z - 1$:

$$\mathcal{L}_{c.t.} = -\frac{1}{4}\delta_A(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a})^2 + i\delta_\psi \bar\psi \not\partial \psi + \delta_c \partial_\mu \bar c_a \partial^\mu c_a - m\delta_m \bar\psi\psi$$

$$- g\delta_{g(A^3)} \partial_\mu A_{\nu a} f_{abc} A^\mu_b A^\nu_c - \frac{1}{4} g^2 \delta_{g^2(A^4)} f_{abe} f_{cde} A_{\mu a} A_{\nu b} A^\mu_c A^\nu_d$$

$$+ g\delta_{g(A\psi^2)} A_{\mu a} \bar\psi \gamma^\mu t_a \psi + g\delta_{g(Ac^2)} f_{abc} \partial_\mu \bar c_a A^\mu_b c_c$$

The structure of the counterterm vertices will be the same as the tree-level ones (and as in QED for the kinetic counterterms).

* Note however that we can write the ratios $g/g_{(s)}$ in several different ways:

$$\frac{g}{g_{(s)}} = Z_{g(A^3)}^{-1} Z_A^{3/2} = Z_{g^2(A^4)}^{-1/2} Z_A = Z_{g(A\psi^2)}^{-1} Z_A^{1/2} Z_\psi = Z_{g(Ac^2)}^{-1} Z_A^{1/2} Z_c$$

this is because one and only g must appear in $\mathcal{L}_{ren}$, to preserve gauge invariance. [More rigorously: Slavnov-Taylor, BRST]

Writing $Z = 1+\delta$ (and further dividing by $Z_A^{1/2}$) one gets

$$\delta_A - \delta_{g(A^3)} = \frac{1}{2}\left(\delta_A - \delta_{g^2(A^4)}\right) = \delta_\psi - \delta_{g(A\psi^2)} = \delta_c - \delta_{g(Ac^2)}$$

this is not as strong as what we got in QED, which was $\delta_\psi = \delta_{e(A\psi^2)}$ (which here would imply that all expressions above vanish).

But recall that in order to show rigorously this identity, we employed the Ward identity, which was based on the fact that $\partial_\mu J^\mu = 0$ with $J^\mu = \bar\psi\gamma^\mu\psi$.

In a non-abelian theory, one can show that $J^\mu_a = \bar\psi\gamma^\mu t_a\psi$ transforms in the adjoint rep:

$\delta J^\mu_a = i f_{abc}\alpha_b J^\mu_c$

Hence the only covariant conservation equation one can write is $D_\mu J^\mu_a = 0$

which implies $\partial_\mu J^\mu_a = i g f_{abc} A_{\mu b} J^\mu_c$

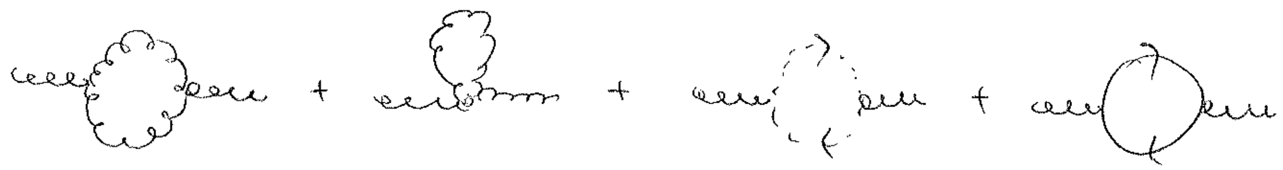
so as soon as $f_{abc} \neq 0$, $\partial_\mu J^\mu \neq 0$ in the theory coupled to the gauge bosons. Hence we cannot conclude that $Z_\psi = Z_g(A\psi^2)$ independently of the gauge fixing.

→ we will need to compute ~~also~~ many δ's to get the β function for g!

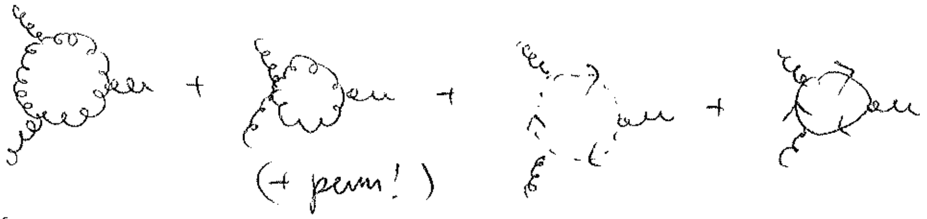
* In order to compute the β-function for g, we can choose any one of the relations between g and $g_{(i)}$. (Indeed, it is instructive to take a look at Gross & Wilczek's original paper!)

Recall that the β-function will be determined by the dependence on the renormalization scale M of the various δ's. Thus we will need to determine the δ for Z_A (for sure), and for any one of the Z_g, and possibly for Z_ψ and/or Z_c.

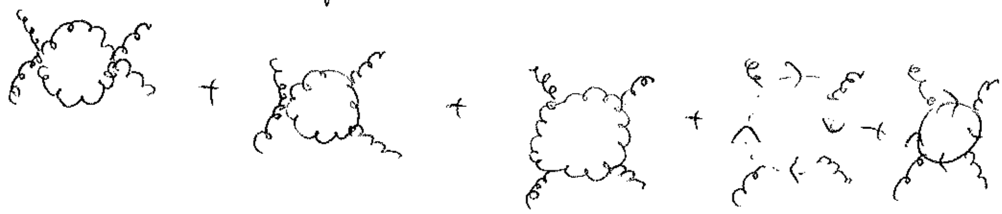
Z_A is determined by the (one-loop) corrections to the $\langle AA \rangle$ propagator:



The diagrams determining $Z_{g(A^3)}$ or $Z_{g^2(A^4)}$ are quite numerous and complicated:

3-pt func
3-gluon vertex: 

(+ perm!)

4-gluon vertex: 

We will thus prefer to compute Z_ψ and $Z_{g(A\psi\bar\psi)}$ or Z_c and $Z_{g(Ac\bar{c})}$ instead:

$\langle \psi\bar\psi \rangle$ corrected by 

A-ψ-$\bar\psi$ vertex corrected by 

$\langle c\bar{c} \rangle$ 

A-c-$\bar{c}$ 

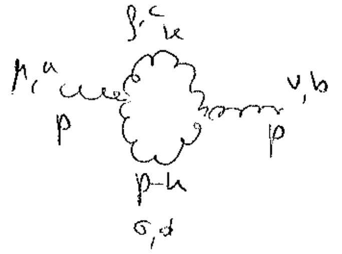
Since ghosts are more intrinsic to the gauge dynamics, they do not involve an arbitrary representation, and do not carry Clifford indices, we will focus on the latter 2 corrections.

* We will thus compute the β-function from

$$g = g_{(0)} Z_A^{1/2} Z_c Z_{g(Ac^2)}^{-1}$$

$$\beta(g) = M\frac{\partial}{\partial M} g = g \, M \frac{\partial}{\partial M}\left(\frac{1}{2}\delta_A + \delta_c - \delta_{g(Ac^2)} \right)$$

In any case we have to compute first δ_A from the correction to the gluon/gauge boson propagator, one diagram at a time.



$$\langle A_1 A_2 \rangle = \frac{1}{2}\langle A_1 A_2 \; i\int g \partial A A_x^2 \; i\int g \, \partial A_y A_y^2 \rangle$$

$$= \frac{1}{2\cdot 6\cdot 6} \langle A_1 A_2 \int g_3 A_x^3 \int g_3 A_y^3 \rangle$$

where g_3 is the 3-gluon vertex $g f_{abc} \eta_{\mu\nu} k_\rho + \dots$

$$\langle A_1 A_2 \rangle = \frac{1}{2\cdot 6\cdot 6}\int g_3^2 \; 2\cdot 3 \langle A_1 A_x \rangle \, 3 \, \langle A_y A_2 \rangle \, 2 \langle A_x A_y \rangle^2$$

so that the 1PI part is $\frac{1}{2}\int_k g_3^2 \langle AA \rangle_k \langle AA \rangle_{p-k}$

we will take the Feynman gauge $\xi = 1$ for all internal propagators. Beware though that the quantities we compute, which are certainly not gauge invariant, do depend on ξ. Of course, the β-function is physical and is gauge invariant, hence any ξ-dependence would eventually cancel.

$$\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{-i}{k^2}\,\frac{-i}{(p-k)^2}\,g^2 f_{acd}\left\{\eta_{\mu\rho}(-k-p)_\sigma+\eta_{\mu\sigma}(2p+k)_\rho+\eta_{\rho\sigma}(2k-p)_\mu\right\}$$
$$\times f_{bcd}\left\{\delta^\rho_\nu(k+p)^\sigma+\delta^\sigma_\nu(-2p+k)^\rho+\eta^{\rho\sigma}(p-2k)_\nu\right\}$$

The group theory part is evaluated as follows:
given a representation ρ of the group g, with generators t^ρ_a, one can define 2 numbers specific to the representation:

$\mathrm{tr}\, t^\rho_a t^\rho_b = T(\rho)\,\delta_{ab}$ where the trace is over the ρ-space.

$t^\rho_a t^\rho_a = C(\rho)\,\mathbb{1}_\rho$ $\quad\mathbb{1}_\rho$ is the identity over ρ-space.

$T(\rho)$ is called the index of ρ, $C(\rho)$ the quadratic Casimir. The fact that one can define in an invariant way $T(\rho)$ and $C(\rho)$ follows from the theory of Lie algebras. They are simply related taking the respective trace over the left-over indices:

$$C(\rho)\dim(\rho)=T(\rho)\dim(G)$$

where $\dim(G)\equiv\dim(Ad)$ the dimension of the adjoint rep.
$\rightarrow$ for Ad, $C(Ad)=T(Ad)$ and we can relate it to f_{abc}:

$(t_a)_{cd}=i f_{cad}$

$\mathrm{tr}\, t^{Ad}_a t^{Ad}_b = C(Ad)\,\delta_{ab} \;\leftrightarrow\; i f_{cad}\, i f_{dbc} = C(Ad)\,\delta_{ab}$

resulting in $f_{acd} f_{bcd} = C(Ad)\,\delta_{ab}$.

We then perform the usual trick of introducing x and write $k = l + px$.

$$+\frac{1}{2} g^2 C(Ad)\delta_{ab} \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2-\Delta)^2} \Big\{ \eta_{\mu\rho} (l + p(1+x))_\sigma + \eta_{\mu\sigma} (l - p(2-x))_\rho + \eta_{\rho\sigma} (p(1-2x) - 2l)_\mu \Big\}$$

$$\times \Big\{ \delta_\nu^\rho (l + p(1+x))^\sigma + \delta_\nu^\sigma (l - p(2-x))^\rho + \eta^{\rho\sigma} (p(1-2x) - 2l)_\nu \Big\}$$

where $\Delta = -p^2 x(1-x)$

In the numerator, we can neglect all terms linear in l since they integrate to 0 . We also take $\eta^{\rho\sigma}\eta_{\rho\sigma} = d$ as we prepare for dimensional regularization.

The numerator becomes

$$\eta_{\mu\nu} \left[\underset{\sim}{2l^2} + p^2\left((1+x)^2 + (2-x)^2\right) \right] + d\left[\underset{\sim}{4 l_\mu l_\nu} + p_\mu p_\nu (1-2x)^2 \right]$$

$$- \underset{\sim}{6 l_\mu l_\nu} + p_\mu p_\nu \left[-2(1+x)(2-x) + 2(1+x)(1-2x) - 2(2-x)(1-2x) \right]$$

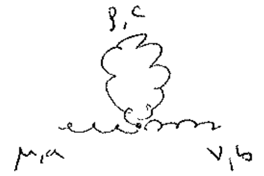
We have a quadratically divergent piece that reads, once we take $l_\mu l_\nu = \frac{1}{d} l^2 \eta_{\mu\nu}$:

$$\frac{1}{2} g^2 C(Ad)\,\delta_{ab} \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{l^2 \eta_{\mu\nu}}{(l^2-\Delta)^2}\, 6\left(1 - \frac{1}{d}\right)$$

The rest is log-divergent so that we can set $d=4$:

$$\frac{1}{2} g^2 C(Ad)\,\delta_{ab} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^2} \left\{ \eta_{\mu\nu} p^2 (5-2x+2x^2) + p_\mu p_\nu (-2-10x+10x^2) \right\}$$

* Before we evaluate the ℓ-integrals, we can already add up the 2 other gauge-sector diagrams:



$$\langle A_1 A_2 \rangle = \langle A_1 A_2 \, i \int g_{4!}^2 A_x^4 \rangle$$

$$= \frac{1}{24} \langle A_1 A_2 \int g_4 A_x^4 \rangle \qquad g_4 = -i g^2 f^2 \eta^2 + \dots$$

$$= \frac{1}{24} \int g_4 4 \langle A_1 A_x \rangle \cdot 3 \langle A_x A_2 \rangle \langle A_x A_x \rangle$$

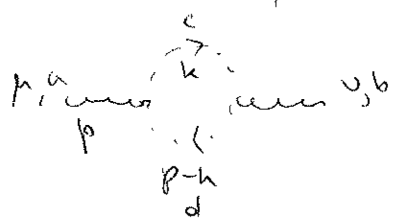
1PI is then $\frac{1}{2} g_4 \int_k \langle AA \rangle_k$

→ we must take the 4-gluon vertex and contract with $\eta^{\rho\sigma}$ and δ^{cd}

$$-i g^2 \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \left\{ f_{ace} f_{bce} (d\eta_{\mu\nu} - \eta_{\mu\nu}) + f_{ace} f_{bce} (d\eta_{\mu\nu} - \eta_{\mu\nu}) \right\}$$

$$= -g^2 C(Ad)\, \eta_{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \frac{d-1}{k^2} \delta_{ab}$$

* We finally have the ghost loop:



$$\langle A_1 A_2 \rangle = \frac{1}{2} \langle A_1 A_2 \, ig \int \partial \bar{c}_x A_x c_x \; ig \int \partial \bar{c}_y A_y c_y \rangle$$

$$= \int \frac{1}{2} 2 \langle A_1 A_x \rangle \langle A_y A_2 \rangle \langle c_x \bar{c}_y \rangle (-1) \langle c_y \bar{c}_x \rangle$$

the 1PI part is $-\int_k g_c \langle c\bar{c} \rangle_{k} g_c \langle c\bar{c} \rangle_{k-p}$ $\quad$ -1 for fermion loop!

$$-g^2 f_{adc} f_{bcd}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2}\,\frac{i}{(p-k)^2}(k-p)_\mu k_\nu$$

$$= -g^2 C(Ad)\,\delta_{ab}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\,\frac{(l-p(1-x))_\mu\,(l+px)_\nu}{(l^2-\Delta)^2}$$

$$= -g^2 C(Ad)\,\delta_{ab}\,\frac{1}{d}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\,\frac{l^2}{(l^2-\Delta)^2}\,\eta_{\mu\nu} \qquad \text{quad-div.}$$

$$+\, g^2 C(Ad)\,\delta_{ab}\,p_\mu p_\nu\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\,\frac{x(1-x)}{(l^2-\Delta)^2} \qquad \text{log-div.}$$

* In order to sum all diagrams, let us rewrite  as follows:

$$-g^2 C(Ad)\,\delta_{ab}\,\eta_{\mu\nu}\int\frac{d^dk}{(2\pi)^d}\,(d-1)\,\frac{(p-k)^2}{k^2(p-k)^2}$$

$$= -g^2 C(Ad)\,\delta_{ab}\,\eta_{\mu\nu}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\,(d-1)\,\frac{l^2+p^2(1-x)^2}{(l^2-\Delta)^2}$$

The quad-div. pieces add up to

$$g^2 C(Ad)\,\delta_{ab}\,\eta_{\mu\nu}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\,\frac{l^2}{(l^2-\Delta)^2}\underbrace{\left[3\left(1-\frac{1}{d}\right)-\frac{1}{d}-d+1\right]}_{-\frac{4}{d}\left(1-\frac{d}{2}\right)^2}$$

and recall

$$\int\frac{d^dl}{(2\pi)^d}\,\frac{l^2}{(l^2-\Delta)^2} = -i\int\frac{d^dl_E}{(2\pi)^d}\,\frac{l_E^2}{(l_E^2+\Delta)^2} = -i\,\frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}}\,\Gamma\!\left(2-\frac{d}{2}\right)\Delta\,\frac{\frac{d}{2}}{1-\frac{d}{2}}$$

so that we get

$$-\frac{ig^2}{(4\pi)^2} C(Ad)\,\delta_{ab}\,\eta_{\mu\nu}\underset{\overset{\shortparallel}{2}}{(d-2)}\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\left(\frac{\Delta}{4\pi}\right)^{\frac{d}{2}-2}\Delta$$

The log-div pieces sum up to, recalling $\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2}=i\frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)$

$$\frac{ig^2}{(4\pi)^2}C(Ad)\,\delta_{ab}\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\Big\{\eta_{\mu\nu}p^2\left[\frac{5}{2}-x+x^2-3+6x-3x^2\right]$$
$$+p_\mu p_\nu\left[-1-5x+5x^2+x-x^2\right]\Big\}\left(\frac{\Delta}{4\pi}\right)^{\frac{d}{2}-2}$$

Now since $\Delta=-p^2x(1-x)$ we finally add the remnant of the quad-div to the log-div, to get

$$\frac{ig^2}{(4\pi)^2}C(Ad)\,\delta_{ab}\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\left(\frac{\Delta}{4\pi}\right)^{\frac{d}{2}-2}\Big\{\eta_{\mu\nu}p^2\left(-\frac{1}{2}+7x-4x^2\right)$$
$$+p_\mu p_\nu\left(-1-4x+4x^2\right)\Big\}$$

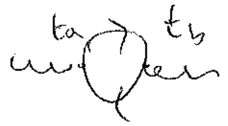
$$=\frac{ig^2}{(4\pi)^2}C(Ad)\,\delta_{ab}\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\left(\frac{\Delta}{4\pi}\right)^{\frac{d}{2}-2}\Big\{(\eta_{\mu\nu}p^2-p_\mu p_\nu)(1+4x-4x^2)\cancel{+\frac{3}{2}\eta_{\mu\nu}p^2(1-2x)}\Big\}$$

odd under $\int dx$!

→ it is transverse!

This is actually an a posteriori justification for neglecting the gauge fixing term in the counter-term Lagrangian. In order to fix δ_A, let us compute the last piece, the fermion loop.

It is actually the same diagram for the self-energy of the photon in QED, modulo the fact that we have to attach a t_a matrix to each vertex.



The diagram will then be proportional to $\mathrm{tr}\, t_a t_b = T(g)\,\delta_{ab}$

$$-\frac{8ig^2}{(4\pi)^2}\,T(g)\,\delta_{ab}\,(p^2\eta_{\mu\nu}-p_\mu p_\nu)\int_0^1 dx\,\Gamma\!\left(2-\frac{d}{2}\right)\left(\frac{\Delta_m}{4\pi}\right)^{\frac{d}{2}-2} x(1-x)$$

$\Delta_m = m^2 - p^2 x(1-x)$, $\Delta_m = \Delta$ if we set $m=0$ (or we neglect it).

We already notice that it has the opposite sign as the pure-~~gauge~~ glue contribution.

* In order to find δ_A, and in particular its Π-dependence, let us note that the kinetic counterterm for A_μ adds up as

$$-i\delta_A(p^2\eta_{\mu\nu}-p_\mu p_\nu)\,\delta_{ab}$$

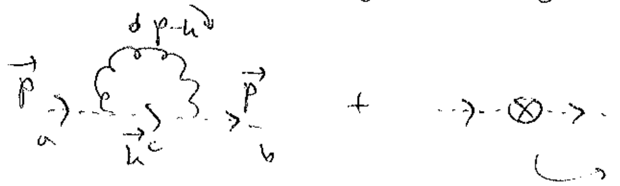
Setting the whole correction to ~~be~~ vanish at $p^2=-\Pi^2$ we find

$$\delta_A = \frac{g^2}{(4\pi)^2}\int_0^1 dx\left\{C(Ad)\left(1+4x(1-x)\right) - T(g)\,8x(1-x)\right\}\left[\frac{1}{\varepsilon}-\gamma-\log\frac{\Pi^2}{4\pi\mu^2}\right]$$

$$=\frac{g^2}{(4\pi)^2}\left(\frac{5}{3}C(Ad)-\frac{4}{3}T(g)\right)\left[\frac{1}{\varepsilon}-\gamma-\log\frac{\Pi^2}{4\pi\mu^2}\right]$$

* We are not done yet, we need to compute δ_c and $\delta_{g(Acc)}$. Let us start from δ_c:

we have a single diagram, and its counter-term



$\hookrightarrow \quad i\delta_c p^2 \delta_{ab}$

$$\langle c_1 \bar{c}_2 \rangle = \frac{1}{2} \langle c_1 \bar{c}_2\, i\int g\, \partial \bar{c}_x A_x c_x \; i\int g\, \partial \bar{c}_y A_y c_y \rangle$$

$$= \frac{1}{\cancel{2}} \int \cancel{2}\, \langle c_1 \bar{c}_x \rangle g_c \langle c_y \bar{c}_2 \rangle g_c \langle A_x A_y \rangle \langle c_x \bar{c}_y \rangle$$

The 1PI part is simply (always $\xi = 1$ gauge!)

$$g^2 f_{dac} f_{dcb} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{(p-k)^2} \frac{i}{k^2} p_\mu k^\mu$$

$$= -g^2 C(Ad)\, \delta_{ab} \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^2}\, p \cdot (\overset{\text{odd}}{\cancel{l}} + px)$$

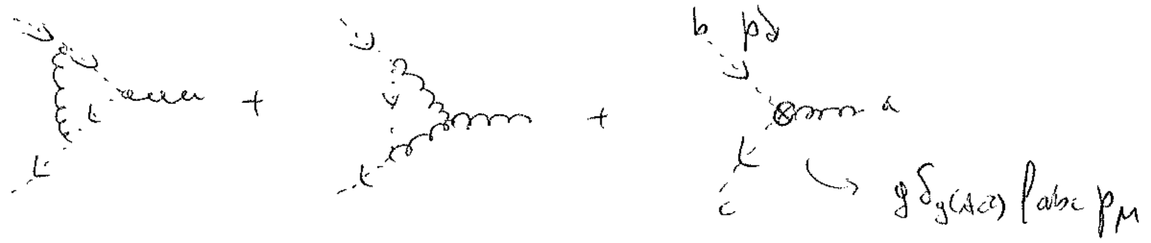
$$= \frac{-ig^2}{(4\pi)^2} C(Ad)\, \delta_{ab}\, p^2 \int_0^1 dx\, x\, \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{\Delta}{4\pi}\right)^{\frac{d}{2}-2}$$

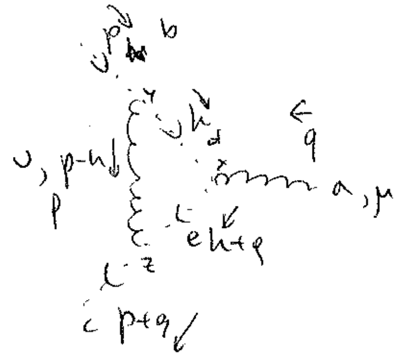
so that we find

$$\delta_c = \frac{g^2}{(4\pi)^2} C(Ad) \int_0^1 dx\, x \left[\frac{1}{\varepsilon} - \gamma - \log \frac{M^2}{4\pi\mu^2} \right]$$

$$= \frac{g^2}{(4\pi)^2} \frac{1}{2} C(Ad) \left[\frac{1}{\varepsilon} - \gamma - \log \frac{M^2}{4\pi\mu^2} \right]$$

× We finally compute $\delta_{g(Ac\bar{c})}$:





$$\langle A_1 c_2 \bar{c}_3\rangle = \frac{1}{6}\langle A_1 c_2 \bar{c}_3 \int g_c A_x \bar{c}_x c_x \int g_c A_y \bar{c}_y c_y \int g_c A_z \bar{c}_z c_z\rangle$$

$$= \frac{1}{6}\int g_c^3\, 3!\, \langle A_1 A_x\rangle\, 2\, \langle c_2 \bar{c}_y\rangle \langle c_z \bar{c}_3\rangle \langle A_y A_z\rangle \langle c_y \bar{c}_x\rangle \times \langle c_x \bar{c}_z\rangle$$

1PI part is :

$$g^3 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{(p-k)^2} \frac{i}{k^2} \frac{i}{(k+q)^2}\, p_\nu (k+q)^\nu k_\mu\, f_{ade} f_{fbd} f_{fec}$$

First of all, the group theory :

$$f_{ade} f_{fbd} f_{fec} = -f_{ade}(i f_{bdf})(i f_{cfe}) = -f_{ade}(t_b^{Ad})_{df}(t_c^{Ad})_{fe}$$

$$= -\frac{1}{2} f_{ade}\left[(t_b^{Ad})_{df}(t_c^{Ad})_{fe} - (t_b^{Ad})_{ef}(t_c^{Ad})_{fd}\right]$$

$$= -\frac{1}{2} f_{ade}\left[(t_b^{Ad})_{df}(t_c^{Ad})_{fe} - (t_c^{Ad})_{df}(t_b^{Ad})_{fe}\right]$$

$$= -\frac{1}{2} f_{ade}\, i f_{bcf} (t_f^{Ad})_{de} = \frac{1}{2} f_{ade} f_{bcf} f_{dfe}$$

$$= -\frac{1}{2} f_{ade} f_{fde} f_{fbc} = -\frac{1}{2} C(Ad) f_{abc}$$

Then we observe that the diagram is log-div.

Hence we can take k very large to compute its diverging part, which is only needed for $\delta_{g(Ac^2)}$:

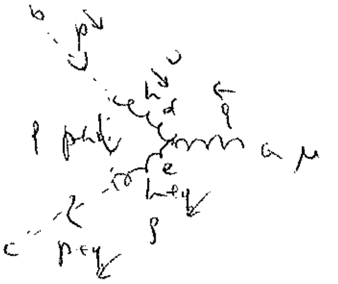
$$-\frac{1}{2} i g^3 C(Ad) f_{abc} \int \frac{d^4k}{(2\pi)^4} \frac{p^\nu k_\nu k_\mu}{(k^2)^3} = -\frac{1}{8} i g^3 C(Ad) f_{abc}\, p_\mu \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2}$$

where we used that $k_\mu k_\nu = \frac{1}{4} k^2 \eta_{\mu\nu}$ under the integral.

After going to Euclidean time and shifting k_E^2 to $k_E^2 + \Delta'$ with a "mock" Δ', we get

$$\frac{1}{8} \frac{g^3}{(4\pi)^2} C(Ad) f_{abc}\, p_\mu\, \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{\Delta'}{4\pi}\right)^{\frac{d}{2}-2}$$

* Similarly we compute the last diagram:



$$(A_1 c_2 \bar{c}_3) = \frac{1}{2} \langle A_1 c_2 \bar{c}_3 \int g_c A_y \bar{c}_y c_y \int g_c A_z \bar{c}_z c_z \int g_3 A_x^3 \rangle$$

$$= \cancel{\tfrac{1}{2}} \int g_c^2 g_3 \langle A_1 A_x \rangle \langle A_y A_x \rangle \langle A_z A_x \rangle \cancel{2} \langle c_2 \bar{c}_y \rangle \langle c_z \bar{c}_3 \rangle \times \langle c_y \bar{c}_z \rangle$$

1PI part is

$$g^3 f_{ade} f_{dbf} f_{efc} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2} \frac{-i}{k^2} \frac{-i}{(k+q)^2}\, p^\nu (p-k)^\rho \left\{ \eta_{\mu\nu}(k-q)_\rho + \eta_{\mu\rho}(2q+k)_\nu + \eta_{\nu\rho}(-2k-q)_\mu \right\}$$

$$= g^3 \left(-\frac{1}{2} C(Ad) f_{abc}\right)(-i) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^3}\, p^\nu (-k^\rho) \left[\eta_{\mu\nu} k_\rho + \eta_{\mu\rho} k_\nu - 2\eta_{\nu\rho} k_\mu \right]$$

taking k large as before

$$= -i\frac{1}{2}g^3 C(Ad) f_{abc} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^3} \left(p_\mu k^2 - \underbrace{k_\mu k_\nu p^\nu}_{\frac{1}{4}k^2 p_\mu} \right)$$

$$= -i\frac{3}{8}g^3 C(Ad) f_{abc}\, p_\mu \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2}$$

$$= \frac{3}{8}\frac{g^3}{(4\pi)^2} C(Ad) f_{abc}\, p_\mu\, \Gamma\left(2-\frac{d}{2}\right)\left(\frac{\Delta'}{4\pi}\right)^{\frac{d}{2}-2}$$

Putting the 2 diagrams together, gives

$$\delta_{g(Ac^2)} = -\frac{1}{2}\frac{g^2}{(4\pi)^2} C(Ad)\left[\frac{1}{\varepsilon} - \gamma - \log\frac{M^2}{4\pi\mu^2}\right]$$ (Note $\delta_c \neq \delta_{g(Ac^2)}$!)

* We are now finally in a position to be able to compute the β-function!

$$\beta = g M \frac{\partial}{\partial M}\left(\frac{1}{2}\delta_A + \delta_c - \delta_{g(Ac^2)}\right)$$

$$\frac{1}{2}\delta_A + \delta_c - \delta_{g(Ac^2)} = \frac{1}{2}\frac{g^2}{2(4\pi)^2}\left[\frac{5}{3}C(Ad) - \frac{4}{3}T(g) + C(Ad) + C(Ad)\right]\left[\frac{1}{\varepsilon} - \gamma - \log\frac{M^2}{4\pi\mu^2}\right)$$

$$= \frac{1}{2}\frac{g^2}{2(4\pi)^2}\left[\frac{11}{3}C(Ad) - \frac{4}{3}T(g)\right]\left(\frac{1}{\varepsilon} - \gamma - \log\frac{M^2}{4\pi\mu^2}\right).$$

This gives:

$$\boxed{\beta = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C(Ad) - \frac{4}{3}T(g)\right)}$$

This is an extremely important result!

* First of all, if there is no matter, i.e in pure Yang-Mills theory, we have $\beta = -\frac{g^3}{(4\pi)^2}\frac{11}{3}C(Ad) < 0$ the coupling decreases towards the UV, i.e. it is <u>asymptotically free</u>.

From group theory, one can show that for $SU(N)$
$C(Ad) = N$, while $T(D) = \frac{1}{2}$

D is the fundamental, N-dimensional rep of $SU(N)$.
Note that ψ being a Dirac spinor, it is actually composed of 2 Weyl spinors, one of which is indeed in the D, while the other is in the $\bar{D}$, the complex conjugate rep. If ρ is a real rep (like Ad), it can be carried by a Weyl spinor.
In any case, in a QCD-like theory with gauge group $SU(N)$ and N_f flavors of quarks, the total matter rep. is N_f copies of D. ($D \oplus \bar{D}$ over Weyl spinors, consistent with the possible presence of a mass term, and the absence of anomalies)

Hence, $T(\rho) = \sum_{N_f} T(D) = \frac{1}{2} N_f$.

Then we have

$$\beta = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right)$$

for N_f sufficiently small, we always have $\beta < 0$.

In real-world QCD, $N=3$ and $N_f = 6$ (at least up to current LHC energies) and $\beta = -7\frac{g^3}{(4\pi)^2} < 0$.

* let us call the group theoretical numerical coefficient b_0, so that

$$\beta = -\frac{g^3}{(4\pi)^2} b_0 \,, \qquad b_0 > 0 \text{ for asymptotic freedom.}$$

We will now take seriously this one-loop value of β and derive physical consequences. Indeed it can be argued that b_0 is observable (and observed), thus it is automatically gauge invariant and scheme independent (i.e. does not depend on the regularization and renormaliz. conditions ~~..~~ – note this is not true for higher loops, except the 2-loop ~~..~~ term).

The equation reads:

$$M\frac{dg}{dM} = -\frac{g^3}{(4\pi)^2} b_0 \qquad \longleftrightarrow \qquad M\frac{d}{dM}\frac{1}{g^2} = \frac{b_0}{8\pi^2}$$

The solution to this simple equation is

$$\frac{1}{g^2(M)} = \frac{1}{g^2(\mu)} + \frac{b_0}{8\pi^2}\log\frac{M}{\mu}$$

if $M > \mu$ $\quad \frac{1}{g^2(M)} > \frac{1}{g^2(\mu)} \iff g^2(M) < g^2(\mu)$ asympt. freedom.

But if we go instead towards the IR ($M < \mu$), we see that there will be an energy scale where the log term cancels the $\frac{1}{g^2(\mu)}$ term, giving a diverging coupling. Let us call this scale Λ_{IR}, for which thus

$$\frac{1}{g^2(\Lambda_{IR})} = 0 = \frac{1}{g^2(\mu)} + \frac{b_0}{8\pi^2}\log\frac{\Lambda_{IR}}{\mu}$$

Plugging this expression in the previous one, or just substituting M for μ, we get the coupling at any scale as a function of that scale and Λ_{IR}:

$$\frac{1}{g^2(M)} = \frac{b_0}{8\pi^2}\log\frac{M}{\Lambda_{IR}} \qquad \text{(provided of course } M > \Lambda_{IR}\text{)}$$

Thus we see that what is intrinsic to a QCD-like theory, is not the coupling, which is scale-dependent, but the IR strong coupling scale Λ_{IR}.

This is called <u>dimensional transmutation</u>.

* Let us turn this expression around. If one is given the coupling g at some scale μ, Λ_{IR} is defined as

$$\Lambda_{IR} = \mu \, e^{-\frac{8\pi^2}{b_0 g^2(\mu)}}$$

Note that Λ_{IR} is a non perturbative quantity: if $g^2(\mu)$ is small, nevertheless there is no way to expand this expression as a power series in g^2. Hence powers of Λ_{IR} will never appear in QCD perturbation theory (only logs, of course). But then Λ_{IR} will be associated to all low-energy, strong coupling phenomena, like generation of a mass gap (glueball and hadronic bound states), confinement (string-like flux-tubes), fermionic bilinear condensates.

* Take real-world QCD: from the data about $\alpha_s = \frac{g_s^2}{4\pi}$ at the electro-weak scale, one can compute $\Lambda_{QCD} \sim 250 \, MeV$ (approximate as we do not trust one-loop β at $\mu \sim \Lambda_{IR}$)

we immediately see that the most familiar QCD bound states, the proton and the neutron, have a mass of the right order of magnitude:

$$m_p \sim m_n \sim 1\ \text{GeV} = O(\Lambda_{QCD}) .$$

(The quark masses, $m_u, m_d \sim$ few MeV, are irrelevant for m_p and m_n)

Similarly, we expect the tension of QCD strings (i.e. flux tubes between confined quarks being pulled apart) to be $T \sim \Lambda^2_{QCD}$.

Finally we expect quark bilinear condensates to form (this breaks spontaneously the dual flavor symmetry of QCD $\to SU(3)$ for the light quarks ~~u~~, u, d and s) with vacuum expectation value $\langle \bar{\psi}\psi \rangle \sim \Lambda^3_{QCD}$.

This is indeed the case, as established from pion physics.