N=1 Supersymmetric Gauge theories

→ Nonperturbative dynamics.

* SUSY action for gauge fields + matter:

$$S = \frac{1}{16\pi} \tau \int d^4x \, d^2\theta \, \mathrm{tr} W^\alpha W_\alpha + \frac{1}{16\pi} \bar{\tau} \int d^4x \, d^2\bar\theta \, \mathrm{tr} \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}$$

$$+ \int d^4x \, d^2\theta \, d^2\bar\theta \; \bar\Phi e^V \Phi + \int d^4x \int d^2\theta \, W(\Phi) + \int d^4x \, d^2\bar\theta \, \bar{W}(\bar\Phi) .$$

gauge sector: $\quad W_\alpha = \bar{D}^2 (e^{-V} D_\alpha e^V)$

$$V = -\theta\sigma^\mu\bar\theta A_\mu + \sqrt{2} i \theta^2 \bar\theta \bar\lambda - \sqrt{2} i \bar\theta^2 \theta \lambda + \theta^2 \bar\theta^2 D$$

(WZ gauge)

$$D_\alpha = \partial_\alpha + \frac{i}{2} \sigma^\mu_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} \partial_\mu \qquad \bar{D}_{\dot\alpha} = \bar\partial_{\dot\alpha} + \frac{i}{2} \theta^\alpha \sigma^\mu_{\alpha\dot\alpha} \partial_\mu$$

so that $\quad W_\alpha = -\sqrt{2} i \lambda_\alpha + \dots \quad - i (\sigma^{\mu\nu})_\alpha{}^\beta \theta_\beta F_{\mu\nu} + \dots$

gauge transformations: $\quad e^V \to e^{-i\bar\Lambda} e^V e^{i\Lambda}$

with Λ a chiral superfield $\quad \bar{D}_{\dot\alpha} \Lambda = 0$

W_α transforms covariantly (as λ_α): $\quad W_\alpha \to e^{-i\Lambda} W_\alpha e^{i\Lambda}$.

Coupling: $\quad \tau = \frac{4\pi}{g^2} - i \frac{\Theta}{2\pi}$

so that $\quad S = \int d^4x \left\{ -\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} + \frac{\Theta}{32\pi^2} \mathrm{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} + \dots$

Matter sector:

Chiral superfields Φ : $\bar{D}_{\dot\alpha} \Phi = 0$

$$\Phi = \phi(y) + \theta\psi(y) + \theta^2 f(y) \qquad y^\mu = x^\mu + \frac{i}{2}\theta\sigma^\mu\bar\theta .$$

Reducible rep. of the gauge group: $\Phi \to e^{-i\Lambda}\Phi$.

$\int d^2\theta d^2\bar\theta\, \bar\Phi e^V \Phi$ gives kinetic terms for ϕ and ψ.

$\int d^2\theta\, W(\Phi) + \text{h.c.}$ gives interaction terms (masses, Yukawa)

Scalar potential: (+ auxiliary fields)

$$\mathcal{L} = \frac{1}{2g^2} D^a D^a + \bar\phi T^a \phi D^a + \bar f^i f_i + \frac{\partial W}{\partial \phi_i} f_i + \frac{\partial \bar W}{\partial \bar\phi^i} \bar f^i$$

eliminate $D^a, f_i, \bar f^i$ by their algebraic EOM:

$$D^a = -g^2 \bar\phi T^a \phi \qquad f_i = -\frac{\partial \bar W}{\partial \bar\phi^i} \quad , \bar f^i = -\frac{\partial W}{\partial \phi_i}$$

so that

$$V = \frac{1}{2} g^2 (\bar\phi T^a \phi)^2 + \frac{\partial W}{\partial \phi_i}\frac{\partial \bar W}{\partial \bar\phi^i} \equiv \frac{1}{2g^2} D^a D^a + f_i \bar f^i$$

Go back to how chiral superfields transform under SUSY transformations

Recall how ordinary translations act: $x \to x+a$

$$\phi(x+a) = e^{-iaP}\phi(x)\, e^{iaP}$$

$$= \phi(x) - i a^\mu [P_\mu, \phi(x)] + \ldots$$

$$= \phi(x) + a^\mu \partial_\mu \phi(x) + \ldots$$

infinitesimal generator: $[P_\mu, \phi(x)] = i\partial_\mu \phi(x) \equiv P_\mu \phi(x)$

SUSY tf: translations in superspace:

$$\theta_\alpha \to \theta_\alpha + \xi_\alpha \quad , \quad \bar\theta_{\dot\alpha} \to \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha}$$

and $$x^\mu \to x^\mu + a^\mu + \underbrace{\frac{i}{2}\theta\sigma^\mu\bar\xi - \frac{i}{2}\xi\sigma^\mu\bar\theta}_{\text{necessary for } \{Q_\alpha, \bar Q_{\dot\alpha}\} = -\sigma^\mu_{\alpha\dot\alpha} P_\mu}$$

we find $$Q_\alpha = \partial_\alpha - \frac{i}{2}\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$$

$$\bar Q_{\dot\alpha} = \bar\partial_{\dot\alpha} - \frac{i}{2}\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

$D_\alpha, \bar D_{\dot\alpha}$ introduced before (with + sign) commute with $Q_\alpha, \bar Q_{\dot\alpha}$, so can be used to define irreducible reps.

How does Φ transform under $Q, \bar{Q}$?

its components satisfy: (by writing $\delta\Phi = (\xi Q + \bar{\xi}\bar{Q})\Phi$ as a chiral superfield)

$$[Q_\alpha, \phi] = \psi_\alpha \qquad [\bar{Q}_{\dot\alpha}, \phi] = 0$$

$$\{Q_\alpha, \psi_\beta\} = -\epsilon_{\alpha\beta} F \qquad \{\bar{Q}_{\dot\alpha}, \psi_\beta\} = -i\sigma^\mu_{\beta\dot\alpha}\partial_\mu\phi$$

$$[Q_\alpha, F] = 0 \qquad [\bar{Q}_{\dot\alpha}, F] = i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\psi^\alpha$$

We learn several things:

- δF is a total derivative $\to$ $\int d^2\theta\, W(\Phi)$ is SUSY invariant
- $\phi = cst$, $\psi = 0 = F$ makes all $[\,,\,] = 0$ $\to$ SUSY vacuum.
- $F \neq 0$ makes $\{Q, \psi\} \neq 0$ $\to$ SUSY broken.

Note $F \neq 0 \to V > 0$

[Similar story for D-terms: $\delta D = \partial_\mu(\ldots)$ and $D \neq 0 \Leftrightarrow$ SUSY broken]

* We now concentrate on the fact that

$[\bar{Q}_{\dot\alpha}, \phi] = 0$ for ϕ lowest component of $\bar{D}_{\dot\alpha}\Phi = 0$.

ϕ is a chiral operator, gauge invariant in a gauge theory.
Given ϕ, one gets ψ and F by acting with Q.

The product of 2 (or more) chiral operators is a chiral operator. (no singularities in the OPE)

In a SUSY vacuum, chiral ops must be constant.
Actually, even correlation fns of chiral ops are constant:

$$\frac{\partial}{\partial x^\mu} \langle 0| \phi(x) \tilde{\Phi}(y) |0\rangle = \langle 0| \frac{\partial}{\partial x^\mu}\phi(x) \cdot \tilde{\Phi}(y) |0\rangle$$

$$= -i \langle 0| [P_\mu, \phi(x)] \tilde{\Phi}(y) |0\rangle$$

$$= \frac{i}{2} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \langle 0| [\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}, \phi(x)] \tilde{\Phi}(y) |0\rangle$$

$$= \frac{i}{2} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \langle 0| \{\bar{Q}_{\dot{\alpha}}, [Q_\alpha, \phi(x)]\} \tilde{\Phi}(y) |0\rangle \quad \left(\begin{array}{l}\text{Jacobi} \\ + \{\bar{Q}_{\dot{\alpha}}, \phi(x)\} = 0\end{array}\right)$$

$$= \frac{i}{2} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \langle 0| \{\bar{Q}_{\dot{\alpha}}, [Q_\alpha, \phi(x)] \tilde{\Phi}(y)\} |0\rangle \quad \left([\bar{Q}_{\dot{\alpha}}, \tilde{\Phi}(y)] = 0 \right)$$

$$= 0 \quad \left(\langle 0| \bar{Q}_{\dot{\alpha}} = 0 = \bar{Q}_{\dot{\alpha}} |0\rangle \right).$$

Thus $\langle \phi(x) \tilde{\Phi}(y) \rangle = \text{cst}$

by cluster decomposition: $\langle \phi(x) \tilde{\Phi}(y) \rangle = \langle \phi \rangle \langle \tilde{\Phi} \rangle$

factorization.

This is true for correlation fns of any number of chiral operators.

Now: since $\langle \{\bar{Q}, \phi\} \rangle = 0$ (in a SUSY vacuum)

if $\phi_1 = \phi_2 + \{\bar{Q}^{\dot{\alpha}}, \psi_{\dot{\alpha}}\}$ then $\langle \phi_1 \rangle = \langle \phi_2 \rangle$

This defines an equivalence class.

⚠ $\psi_{\dot{\alpha}}$ gauge invariant!

In superfields, this writes

$$\Phi_1 = \Phi_2 + \bar{D}^{\dot\alpha} \Sigma_{\dot\alpha}$$

Note that for Φ_1 to be chiral ($\bar{D}_{\dot\alpha} \Phi_1 = 0$)

we have to have $\bar{D}_{\dot\beta} \bar{D}^{\dot\alpha} \Sigma_{\dot\alpha} = 0$.

it is straightforward to show that for a general superfield $\Sigma_{\dot\alpha}$ (satisfying the constraint above) the lowest component of $\bar{D}^{\dot\alpha} \Sigma_{\dot\alpha}$ is an operator which can be written as $\{\bar{Q}^{\dot\alpha}, \psi_{\dot\alpha}\}$. (essentially $\bar{D}_{\dot\alpha} \sim \bar{Q}_{\dot\alpha}$).

We can actually always find a Z such that

$$\Phi_1 = \Phi_2 + \bar{D}^2 Z .$$

Relations in the chiral ring:

Example: EOM in WZ model:

$$\bar{D}^2 \bar\Phi = \frac{\partial W}{\partial \Phi} \quad \Rightarrow \quad \text{in a SUSY vacuum } \left\langle \frac{\partial W}{\partial \Phi} \right\rangle = 0 .$$

Gauge theories: a tricky one is the glueball (chiral) superfield

$$S = -\frac{1}{32\pi^2} \operatorname{tr} W^\alpha W_\alpha$$

it can be written as $S \sim \bar{D}^2 \operatorname{tr}(e^{-V} (D_\alpha e^V) W^\alpha)$

does this mean $\langle S \rangle = 0$?

No because $\operatorname{tr}(e^{-V} D_\alpha e^V W^\alpha)$ is <u>not</u> gauge invariant.

In other words, the lowest component of S is:

$$\mathrm{tr}\lambda\lambda \sim \{ \bar{Q}, \mathrm{tr}\lambda A_\mu \}$$

but $\mathrm{tr}\lambda A_\mu$ not gauge invariant. → the VEV of a gauge invariant object ~~(…)~~, being unobservable, need not be SUSY invariant.

* Upshot: scalar chiral operators can have constant VEVs in a SUSY vacuum. → they will be important to characterize such vacua.

Recall $\mathcal{V} = \frac{1}{2g^2} D^a D^a + f_i \bar{f}^i \geq 0$

SUSY vacuum $\Leftrightarrow$ $D^a = 0$ $f_i = 0 = \bar{f}^i$ $\Leftrightarrow$ $\mathcal{V} = 0$

So: SUSY vacua are determined by the equations:

$D^a = 0$ (D-flatness) $\Leftrightarrow$ $\bar{\phi} T^a \phi = 0$

$f^i = 0$ (F-flatness) $\Leftrightarrow$ $\frac{\partial W}{\partial \phi_i} = 0$

if $W = 0$, we are left with $D^a = 0$
which leads to the classical moduli space of a gauge theory.
≡ Manifold of physically inequivalent SUSY vacua, parametrized by some scalar VEVs of chiral ops.

For a gauge group G, there are $\dim G$ real conditions $D^a = 0$, which constrain the VEVs ϕ_i can have.

Moreover, we have to mod out (≡ equivalence classes) by the gauge transformations.

The moduli space $\mathcal{M}$ is given by the equivalence classes under G of ϕ_i satisfying $D^a=0$: $\mathcal{M} = \{\phi_i \mid D^a = 0\}/G$

By using the complex gauge (relaxing the WZ gauge on V) it is possible to show that, given a set of values ϕ_i, the orbit under $G_{\mathbb{C}}$ (complexified → non compact gauge group) always intersects once the submanifold $D^a=0$.

Hence another way to characterize $\mathcal{M}$ is by

$$\mathcal{M} = \{\phi_i\}/G_{\mathbb{C}}$$

Now: gauge invariant chiral operators are naturally $G_{\mathbb{C}}$ invariant → $\mathcal{M}$ is naturally parametrized by such operators. (polynomial in ϕ_i so as not to have spurious singularities)

There might be relations among the operators, some algebraic and some implied by $f_i=0$ when $W \neq 0$.

<u>Note</u>: $\mathcal{M}$ is <u>not</u> parametrized (only) by Goldstone bosons. Actually, $\mathcal{M}$ is typically non compact. If there are Goldstone bosons, in a SUSY vacuum they have a scalar partner, typically non compact, unrelated to any broken global symmetry. Unlike non SUSY field theories, these non compact flat directions are protected.

* As just said, global symmetries are important to characterize the theory.

Global symmetries can be non abelian:

$$\phi_i \to M^i{}_j \phi_i \qquad \text{with } M \text{ a (real) rep. of a}$$

group G_{global}.

Any chiral superfield, being complex, has a $U(1)$ symmetry rotating it by a phase: $\Phi \to e^{i\alpha}\Phi$, $\bar{\Phi} \to e^{-i\alpha}\bar{\Phi}$ so that $\bar{\Phi}\Phi$ is invariant. $W(\Phi)$ breaks such a symmetry, except if one gives charges to the couplings.

There is a special $U(1)$ symmetry specific to SUSY, it is the R-symmetry which rotates $\theta \to e^{i\alpha}\theta$ $\bar{\theta} \to e^{-i\alpha}\bar{\theta}$. Note that $\int d^2\theta \to e^{-2i\alpha}\int d^2\theta$ so W has to have R-charge 2 to preserve R-symmetry.

Note that under $U(1)_\Phi$ if $\Phi = \phi + \theta\psi + \theta^2 f$

$$\phi \to e^{i\alpha}\phi, \; \psi \to e^{i\alpha}\psi, \; f \to e^{i\alpha} f$$

while under $U(1)_R$ if, say, Φ has R-charge 0

$$\phi \to \phi \qquad \psi \to e^{-i\alpha}\psi \qquad f \to e^{-2i\alpha} f.$$

every component transforms differently.

Most notably ψ has R-charge $R_\Phi - 1$.

* Non renormalization of $W(\phi)$.

A simple argument based on symmetries.

Take the WZ model of a single chiral superfield Φ

with $W(\Phi) = m\Phi^2 + g\Phi^3$.

Take the following charge assignments:

	Φ	m	g	W
$U(1)_\phi$	1	-2	-3	0
$U(1)_R$	0	2	2	2

Then any correction must be a holomorphic fn of m, g, Φ which has $U(1)_\phi$ charge 0 and $R=2$.

write: $W(\Phi) = m\Phi^2 f(m, g, \Phi)$.

f has ϕ- and R-charge 0.

R-charge 0: $\frac{g}{m}$ ϕ-charge -1

$\to$ only invariant is $\Phi\frac{g}{m}$

$$W(\Phi) = m\Phi^2 f\left(\frac{g}{m}\Phi\right).$$

when $g\to 0$ it must be well behaved

and also when $m\to 0$

$\to$ it can only be $f(t) = 1+t$.

$$\to W(\Phi) = m\Phi^2 + g\Phi^3.$$

* Less esoterically, it can be shown that perturbative UV divergences can only generate D-term corrections i.e. terms like $\int d^2\theta d^2\bar\theta\, F(\Phi,\bar\Phi)$.

Note that these ~~...~~ terms lead to a renormalization of the ~~...~~ kinetic term

$$\int d^2\theta d^2\bar\theta\, \bar\Phi\Phi \rightarrow \int d^2\theta d^2\bar\theta\, K(\Phi,\bar\Phi) \qquad K: \text{Kähler potential.}$$

This is just wave fn renormalization. The physical couplings will be renormalized because of that.

Also, there can be (and there are indeed) singular D-term corrections such as

$$\int d^2\theta d^2\bar\theta\, \Phi \frac{D^2}{\Box} \Phi \simeq \int d^2\theta\, \Phi^2 \qquad \left(\text{because } \int d^2\bar\theta \equiv \bar D^2 \text{ and } \bar D^2 D^2 = \Box\right)$$

but these are IR effects when there are massless particles.

In a Wilsonian scheme, where we integrate over energies down to some scale μ, these IR divergences are not there and the effective superpotential is always holomorphic.
We will always refer to this Wilsonian effective superpotential.

Just remember that there are IR effects + wave fn renorm. to go back to physical quantities.

<u>Note</u>: Wilson $\equiv$ 1PI if all particles massive. (as in example before)

So, since there are no perturbative corrections to W, we write for the low-energy effective superpotential, in the Wilsonian scheme:

$$W_{eff} = W_{tree} + W_{non\text{-}perturbative}$$

$W_{non\text{-}pert}$ cannot be excluded by ~~a~~ perturbative non renorm. theorem, but can be constrained by symmetries. E.g. $W_{np}=0$ in WZ model above.

* Renormalization of gauge coupling.

We know from non SUSY gauge theories that it is renormalized.

However: τ appears holomorphically in an F-term:

$$\mathcal{L} = -\int d^2\theta \frac{i\tau}{\cancel{4\pi}} S + h.c. \qquad S = -\frac{1}{32\pi^2} \mathrm{tr}\, W^\alpha W_\alpha$$

no renormalization?

Does not apply here because we can generate the D-term

$$\int d^2\theta\, d^2\bar\theta\, \mathrm{tr}\left(e^{-V} D^\alpha e^{V} \cdot W_\alpha\right)$$

The integrand is not gauge invariant (the integral is) but local. $\rightarrow$ τ can indeed run.

Basically, $\int d^2\theta\, \tau S$ is a special F-term because it is the kinetic term for the gauge field. (and gaugino)

Unlike $\int d^2\theta\, W(\Phi)$ which do not contain derivatives of Φ.

Nevertheless, holomorphy still applies.
How can τ vary with the scale of renormalisation μ?

$$\mu \frac{d\tau}{d\mu} = \frac{1}{2\pi}\beta(\tau).$$

but $\tau = \frac{4\pi}{g^2} - i\frac{\Theta}{2\pi}$

and from ordinary field theory arguments we know that

$$\mu\frac{d\Theta}{d\mu} = 0 \qquad \text{and} \qquad \mu\frac{d}{d\mu}\left(\frac{1}{g^2}\right) = \beta\left(\frac{1}{g^2}\right).$$

This would lead to β a real fn of $\mathrm{Re}\,\tau$.
→ clearly not holomorphic.
Only choice of β compatible with holomorphicity is $\beta = \text{cst}$, which can be a real cst.

$$\mu\frac{d\tau}{d\mu} = \frac{1}{2\pi}\beta \qquad \to \qquad \mu\frac{d}{d\mu}\left(\frac{4\pi}{g^2}\right) = \frac{1}{2\pi}\beta$$

or $\mu\frac{dg}{d\mu} = -\frac{\beta}{16\pi^2}g^3$ <u>one-loop contribution!</u>

β-fn is exact at one-loop. This is of course not true in a standard scheme, but it is true in the Wilsonian scheme where β stays holomorphic. In other schemes (such as NSVZ) β is not holomorphic, there are higher loop contributions, but this is related to IR effects.

From now on, we consider only the Wilsonian τ, which is renormalized only to one-loop.

One very important quantity is the RG invariant scale Λ, which is holomorphic here:

$$\mu \frac{d\tau}{d\mu} = \frac{\beta}{2\pi} \quad \longrightarrow \quad \tau(\mu) = \frac{\beta}{2\pi} \ln \frac{\mu}{\Lambda}$$

$$\Lambda^\beta = \mu^\beta e^{-2\pi\tau(\mu)} = \mu^\beta e^{-\frac{8\pi^2}{g^2(\mu)} + i\Theta}$$

$|\Lambda|$ is just as in Λ_{QCD} : when $\mu \to \Lambda$ from above $g \to \infty$

the phase of Λ^β is interesting because it involves Θ, which can be shifted by anomalies.

Moreover Λ^β is just e^{-S} of one instanton.

* What is the constant β?

$$\beta = \frac{1}{2}\left(3\,T(Adj) - \sum_r T(r)\right)$$

$T(Adj)$ is index of adjoint rep, due to gauge sector.

$T(r)$ index of reps of matter superfields.

for $SU(N_c)$ $\quad T(Adj) = 2N_c \quad T(fund) = 1 = T(\overline{fund})$

So if there are N_f $Q, \tilde{Q}$ in $fund \oplus \overline{fund}$

$$\beta = 3N_c - N_f \quad . \quad \beta > 0 \text{ asymptotic freedom.}$$

* We will now concentrate on determining the low-energy (Wilsonian) effective superpotential. Its interest is in determining the properties of the SUSY vacua of the theory. Assuming there is no relevant low energy gauge sector (confinement), its extrema give the SUSY vacua.

Generically, we have a tree level superpotential

$$W_{tree} = \sum_\alpha \lambda_\alpha X_\alpha \qquad \text{with } X_\alpha(\Phi_i) \text{ gauge invariants.}$$

by perturbative non renormalization we expect:

$$W_{eff} = W_{tree} + W_{non\,pert.}$$

What does $W_{non\,pert}$ depend on?

it could depend on X_α, Λ and λ_α.

Linearity principle: it does not depend on λ_α.

As already stated, λ_α cannot appear in a perturbative series. Non perturbative contributions, if they are holomorphic, do not have a sensible $\lambda \to 0$ limit.

[it really is a conjecture, but with a lot of evidence and no counter examples).

$$\text{Thus} \qquad W_{eff} = \sum_\alpha \lambda_\alpha X_\alpha + W_{non\,pert}(X_\alpha, \Lambda)$$

Symmetries can sometimes fix $W_{n.p.}$ up to a constant.

* Integrating out. This is the essence of the Wilsonian RG flow.

Two simple cases are: i) when there are some flavors N_f' with masses m, we know that the theory for $\mu > m$ has $\beta = 3N_c - N_f$, while for $\mu < m$ we have a larger β-fn: $\tilde{\beta} = 3N_c - (N_f - N_f')$.

The matching of the dynamical scales is as follows:

$$\tilde{\Lambda}^{\tilde{\beta}} = \Lambda^{\beta} m^{N_f'} \quad \text{(continuity of one loop RG flow)}$$

ii) Some massless flavors N_f' have VEVs which break $SU(N_c)$ to $SU(N_c - N_f')$ at $\mu \sim \langle\phi\rangle$.

For $\mu > \langle\phi\rangle$ we have $\beta = 3N_c - N_f$

For $\mu < \langle\phi\rangle$ $\quad \tilde{\beta} = 3(N_c - N_f') - (N_f - N_f') = 3N_c - N_f - 2N_f'$

↳ flavors are eaten.

$$\tilde{\Lambda}^{\tilde{\beta}} = \frac{\Lambda^{\beta}}{\langle\phi\rangle^{2N_f'}}$$

Integrating out can be done by solving the classical EOM of the fields with $M > \mu$.

As this EOM essentially involve scalars, the only non trivial piece is algebraic and involves extremizing W_{eff}.

In particular, suppose $\lambda_1 X_1$ is a mass term for some elementary fields ϕ_i → integrating them out is done by:

$$\frac{\partial}{\partial X_1} W_{eff} = 0 \qquad \rightarrow X_1 = X_1(\lambda_1, \dots)$$

because of the structure of W_{eff}, this becomes:

$$\lambda_1 + \frac{\partial}{\partial X_1} W_{m.p.} = 0 \qquad \text{Legendre transform!}$$

If we integrate out all X_α and substitute back, we obtain $W_{eff}(\lambda_\alpha, \Lambda)$ which is Legendre tf of $W_{mp}(X_\alpha, \Lambda)$.

This suggest 2 things: first, we can easily compute VEVs of X_α by recollecting the linear expression of W_{eff}:

$$\langle X_\alpha \rangle = \frac{\partial}{\partial \lambda_\alpha} W_{eff} \qquad \text{. This remains true when } W_{eff}(\lambda_\alpha, \Lambda).$$

Second: we can even use the above expression to express back λ_α in terms of X_α and reobtain $W_{mp}(X_\alpha, \Lambda)$!! Inverse Legendre transform.

This is "integrating in": against Wilsonian wisdom, but follows from linearity principle $\equiv$ holomorphy and non renormalisation.

* First example: Pure $SU(N_c)$ Super Yang-Mills

At low energy, we assume the theory confines and the dynamics is governed by the glueball superfield

$S \propto \mathrm{tr}\lambda\lambda + \dots$ (gluino bilinear)

What are the global symmetries of the theory?

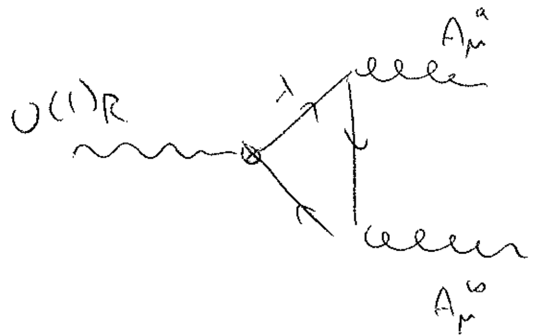
We only have R-symmetry: $U(1)_R : \theta \to e^{i\alpha}\theta$.

Remember $W_\alpha = \lambda_\alpha + \theta^\beta F_{\alpha\beta} + \dots$

$F_{\mu\nu} \to F_{\mu\nu}$ under $U(1)_R \Rightarrow \lambda_\alpha \to e^{i\alpha}\lambda$

and thus $S \to e^{2i\alpha} S$ under $U(1)_R$.

⚠ $U(1)_R$ is like an axial symmetry for the Weyl fermions λ.



$\sim \mathrm{tr}_{adj} T^a T^b \sim 2N_c \delta^{ab}$

Say, in the path integral, one can see that a $U(1)_R$ rotation of λ is accompanied by a shift of the action:

$$\mathcal{L} \to \mathcal{L} - \frac{2N\alpha}{32\pi^2} \mathrm{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

(which can be ~~compensated~~ absorbed by $\Theta \to \Theta + 2N\alpha$, from which we derive that $U(1)_R$ is broken to $\mathbb{Z}_{2N}$)

In SUSY terms:

$$\mathcal{L} \to \mathcal{L} - 2N\alpha i \int d^2\theta\, S + c.c.$$

At the effective level, we want $S \to S e^{2i\alpha}$ to reproduce the same shift:

$$\mathcal{L}_{eff} = \dots - \int d^2\theta\, N_c S \log \frac{S}{\mu^3} + c.c.$$

Note that if $\Theta \to \Theta + 2N\alpha$ (+ one loop), this implies $\Lambda^{3N_c} \to \Lambda^{3N_c} e^{2Ni\alpha}$

and the tree level Λ term compensates for the shift:

$$\mathcal{L}_{tree} = -2\pi i \int d^2\theta\, \tau\, S + c.c = \int d^2\theta\, 3N_c \log \frac{\Lambda}{\mu} \cdot S + c.c.$$

Collecting the terms, we write

$$W_{eff} = - S \log \left(\frac{S}{\Lambda^3}\right)^{N_c} + N_c S$$

this term added for convenience by a finite rescaling of Λ.

We can now extremize this $W_{eff}(S)$ and eliminate S (which is massive anyway).

$$\frac{\partial}{\partial S} W_{eff} = 0 \iff \log\left(\frac{S}{\Lambda^3}\right)^{N_c} = 0 \iff \langle S^{N_c} \rangle = \Lambda^{3N_c}$$

$c=1$

Λ^{3N_c} is a one-instanton contribution! ($\exists$ explicit computation)

$= \mu^{3N_c} e^{-\frac{8\pi^2}{g^2(\mu)} + i\Theta}$

Moreover, by factorization we learn that $\langle S\rangle \neq 0$

$$\langle S\rangle \propto \langle \mathrm{tr}\,\lambda\lambda\rangle \propto \Lambda^3 e^{\frac{2\pi i k}{N_c}} \qquad k=0\ldots N_c-1 .$$

$\langle S\rangle$ breaks the non anomalous $\mathbb{Z}_{2N}$ R-symmetry to $\mathbb{Z}_2$.

→ Chiral symmetry breaking.

there are consequently N_c vacua, corresponding to the N_c-th roots of unity labeling the different $\langle S\rangle$.

Also: $W_{eff}(\Lambda) = N_c \Lambda^3 e^{\frac{2\pi i k}{N_c}}$

there are domain walls between the $\neq$ vacua, with

$$T \sim |\Delta W_{eff}| \sim \Lambda^3 .$$

An important thing to remember is that $W_{eff} = N_c\Lambda^3$ for $SU(N_c)$ SYM. Indeed often this is the theory we get at very low energies, then by matching of scales one can obtain $W_{eff}(X,\Lambda)$ quite easily.

We now finally turn to SQCD.

What is Supersymmetric QCD.

It is a gauge theory with gauge group $SU(N_c)$ and with N_f pairs of chiral superfields $Q_i, \tilde{Q}^i$ in the $\square \oplus \bar{\square}$ of $SU(N_c)$.

E.g. we can take gauge fields A_μ, gaugino λ_α and superfields V, W_α to be $N_c \times N_c$ matrices. (traceless) $(W_\alpha)^a{}_b$ and $(Q_i)^a$ to be column vectors while $(\tilde{Q}^i)_a$ are row vectors.

This is a vectorial theory: $(\square \oplus \bar{\square})^* = \square \oplus \bar{\square}$.

With respect to real world QCD it has the gaugini in the adjoint of $SU(N_c)$, and the squarks which are the lowest components of Q, $\tilde{Q}$: $\quad Q = q + \theta \psi_Q + \ldots$

So it really is QCD + scalars.

The action is:

$$\mathcal{L} = \frac{\tau}{16\pi} \int d^2\theta \, \mathrm{tr}\, W^\alpha W_\alpha + h.c.$$

$$+ \int d^2\theta d^2\bar{\theta} \left(Q^{\dagger i} e^V Q_i + \tilde{Q}^\dagger_i e^{-V} \tilde{Q}^i \right)$$

$$+ \int d^2\theta \, W(Q, \tilde{Q}) + h.c.$$

Only renormalizable term in W is:

$$W(Q, \tilde{Q}) = m^i{}_j Q^a_i \tilde{Q}^j_a$$

Global symmetries:

we have a $U(N_f)$ rotating Q_i and another $U(N_f)$ rotating $\tilde{Q}^i$, and an R-symmetry

All in all we have:

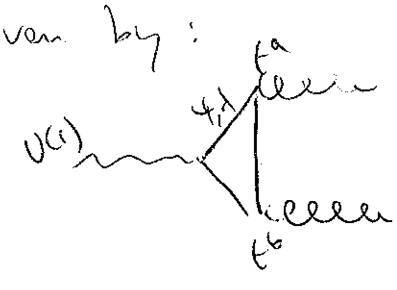
	$SU(N_f)$	$\widetilde{SU}(N_f)$	$U(1)_B$	$U(1)_A$	$U(1)'_R$
Q	$\bar{N}_f$	1	1	1	1
$\tilde{Q}$	1	N_f	-1	1	1

Note that $\psi_q, \tilde{\psi}_q$ have R-charge 0, while λ_α has R-charge 1.

The R-charge is actually a choice, it can be shifted using $U(1)_B$ and $U(1)_A$.

We ~~will~~ can see that ~~a combination of~~ $U(1)_A$ and $U(1)'_R$ are anomalous, but a combination is anomaly free, which we will call $U(1)_R$.

The contribution to ~~any~~ the anomaly of any $U(1)$ is given by:



$\propto Q(\psi) \, \mathrm{Tr}\, t^a t^b$

↳ normalized to 1 for □

As we said the anomaly can be reabsorbed by a ~~charge~~ shift of Θ, or equivalently a phase rotation of Λ^β

Note: $\beta = 3N_c - N_f$

charges of $\Lambda^{3N_c-N_f}$ are the following:

under any $U(1)_Q$: $N_f\left(Q(\psi_Q)+Q(\tilde{\psi}_Q)\right)+2N_c\,Q(\lambda)$

$U(1)_B$: 0

$U(1)_A$: $2N_f$ } anomalous.

$U(1)'_R$: $2N_c$

but there is a non anomalous combination of $U(1)'_R$ and $U(1)_A$, that we call $U(1)_R$:

$$Q_R = Q'_R - \frac{N_c}{N_f}\,Q_A\,.$$

Under $U(1)_R$ λ has charge 1 (as θ)

$Q,\tilde{Q}$ have charge $\frac{N_f-N_c}{N_f}$

$\psi_Q,\tilde{\psi}_Q$ have charge $-\frac{N_c}{N_f}$.

$\Lambda^{3N_c-N_f}$ has charge 0.

A mass term $W=m^i_{\ j}\,Q_i\,\tilde{Q}^j$

typically breaks some global symmetries ($SU(N_f)\times\widetilde{SU}(N_f)\times U(1)_R$)

but we can also let $m^i_{\ j}$ transform as $(\Box,\bar{\Box})$ of $SU(N_f)\times\widetilde{SU}(N_f)$

and give it an ~~$2\frac{N_f-N_c}{N_f}$~~ R-charge $= 2\frac{N_c}{N_f}$

* Classical moduli space

Let us solve some D-flatness conditions. For the moment we set $W=0$.

From the kinetic term $\int d^2\theta\, d^2\bar{\theta}\, \left(Q^\dagger e^V Q + \tilde{Q}^\dagger e^{-V} \tilde{Q}\right)$

we get the D-terms:

$A = 1 \ldots N_c^2-1$

$$Q^{\dagger i}_a T^A Q_i - \tilde{Q}^i T^A \tilde{Q}^\dagger_i = 0 \qquad \left(\text{note that } T^A_{Q} = -T^A_{\bar{Q}}\right)$$

T^A: basis of hermitian traceless matrices

we can write:

$$Q^{\dagger i}_a Q^b_i - \tilde{Q}^i_a \tilde{Q}^{\dagger b}_i = k\, \delta^b_a \quad , \quad k = \frac{1}{N_c}\left(Q^{\dagger i}_a Q^a_i - \tilde{Q}^i_a \tilde{Q}^{\dagger a}_i\right)$$

* Let us first consider the case $N_f < N_c$.

$Q^{\dagger i}_a Q^b_i$ is an $N_c \times N_c$ matrix of rank $N_f < N_c$.

by $SU(N_c)$ rotations we can put it in a diagonal form, with the last $N_c - N_f$ entries equal to zero.

This amounts to having

$$Q^b_i = \begin{array}{c} \overbrace{\qquad\qquad}^{N_f} \\ \left[\begin{array}{ccc} q_1 & & 0 \\ & \ddots & \\ 0 & & q_{N_f} \\ & 0 & \end{array}\right] \end{array} \Bigg\} N_c$$

Then, since $Q^\dagger Q$ is diagonal, and also $k\delta$

$\rightarrow$ $\tilde{Q}\tilde{Q}^\dagger$ must also be diagonal.

Since it is also of rank $N_f < N_c$, $N_c - N_f$ entries are zero.

non zero entries of both $Q^\dagger Q$ and $\tilde{Q}\tilde{Q}^\dagger$ are positive $\sim |q|^2, |\tilde{q}|^2$

Then it is clear that the only way to satisfy the D-eqs is to have $k=0$ and $\tilde{q}\tilde{q}^\dagger$ with non zero entries in the first N_f positions. Equivalently:

$$\tilde{q}^i_a = \underbrace{\begin{bmatrix} \tilde{q}_1 & & 0 & & \\ & \ddots & & & 0 \\ 0 & & \tilde{q}_{N_f} & 0 & \cdots \end{bmatrix}}_{N_c} \Big\} N_f$$

and with: $\qquad |q_i|^2 = |\tilde{q}_i|^2 \quad \forall i = 1..N_f$.

This is of course one representative in a gauge equivalence class of VEVs which satisfy the D-flatness conditions.

One important conclusion is that such generic VEVs break the gauge group from $SU(N_c)$ to $SU(N_c-N_f)$. (of course there are pts of enhanced gauge symmetry).

How to parametrize the moduli space by gauge invariants? Let us first guess $\dim \mathcal{M}$ by a simple argument.

We start with N_c^2-1 vector multiplets and $2N_fN_c$ chiral multiplets.

we end up with $(N_c-N_f)^2-1$ vector multiplets of the unbroken gauge group.

The other $N_c^2-1-(N_c-N_f)^2+1 = 2N_cN_f - N_f^2$ vector multiplets have become massive, by eating the same amount of chiral multiplets (in $\mathcal{N}=1$, massive vector multiplet has 3 d.o.f of the massive vector + 1 real scalar $\to$ 2 more w.r.t. the massless vector multiplet)

We are left with $2N_f N_c - (2N_f N_c - N_f^2) = N_f^2$
massless chiral multiplets, singlets of $SU(N_c - N_f)$.

Thus $\dim_{\mathbb{C}} \mathcal{M} = N_f^2$

Gauge invariants: $M_i{}^j = Q_i^a \tilde{Q}_a^j$

meson superfield (note however that lowest component which gets a VEV is a bilinear in the squarks)

* What changes when $N_f \geq N_c$?

We see that there are more invariants: we can now use the $\varepsilon_{a_1 \ldots a_{N_c}}$ tensor of $SU(N_c)$ $\rightarrow$ baryons.

Global symmetries broken at generic pt of moduli space: $SU(N_f) \times \widetilde{SU}(N_f)$ broken to a diagonal subgroup, and $U(1)_R$ $\rightarrow$ we expect $N_f^2 - 1 + 1 = N_f^2$ Goldstone bosons (real)
Hence, we can think of all mesons as having a compact component which is a Goldstone boson, while its non compact partner is a massless pseudo Goldstone boson which remains massless due to SUSY.

If we rewrite the D-flatness conditions:

$$Q^{+i}_a Q^b_i - \tilde{Q}^i_a \tilde{Q}^{+b}_i = k \delta^a_b$$

Now both $Q^+ Q$ and $\tilde{Q}\tilde{Q}^+$ are rank N_c matrices

(the sum runs for $i = 1 \ldots N_f \geq N_c$)

We can actually use $SU(N_f)$ to put Q_i^b in a simple form (see them as N_c N_f-vectors)

$$Q_i^b = \underbrace{\left[\begin{array}{ccc|c} q_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & q_{N_c} & \end{array}\right]}_{N_f} \Big\} N_c$$

Similarly, we can use $\widetilde{SU}(N_f)$ to get:

$$\tilde{Q}_a^i = \underbrace{\left(\begin{array}{ccc} \tilde{q}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{q}_{N_c} \\ \hline & 0 & \end{array}\right)}_{N_c} \Big\} N_f$$

Then the D-flatness eqs are $|q_i|^2 - |\tilde{q}_i|^2 = k \quad \forall i$.

with $k = \frac{1}{N_c} \sum_i \left(|q_i|^2 - |\tilde{q}_i|^2\right)$

$\rightarrow$ we can now have $k \neq 0$.

The relation to baryons is that we can set $\tilde{q}_i = 0$ and all $q_i = q$ $\quad (k = |q|^2)$.

Then all mesons vanish: $M_i^j = 0$ but

$B_{i_1 \ldots i_{N_c}} = q^{N_c} \neq 0$, where we have defined the baryons

	$U(1)_B$	$U(1)_R$
$B_{i_1 \ldots i_{N_c}} = \varepsilon_{a_1 \ldots a_{N_c}} Q_{i_1}^{a_1} \ldots Q_{i_{N_c}}^{a_{N_c}}$	N_c	$\frac{N_c}{N_f}(N_f - N_c)$
$\tilde{B}^{i_1 \ldots i_{N_c}} = \varepsilon^{a_1 \ldots a_{N_c}} \tilde{Q}_{a_1}^{i_1} \ldots \tilde{Q}_{a_{N_c}}^{i_{N_c}}$	$-N_c$	$\frac{N_c}{N_f}(N_f - N_c)$

We have many invariants: $M_i^{\ j} \to N_f^2$

$$B, \tilde{B} \to 2 \cdot \frac{N_f!}{N_c!(N_f-N_c)!}$$

However $\dim_{\mathbb{C}} \mathcal{M}$ is given by:

at a generic pt, all of gauge symmetry is broken

$\to N_c^2-1$ dual superfields are eaten.

we are left with $2N_fN_c - N_c^2+1$ dual superfields. Much less!

* e.g. $N_f = N_c$: N_c^2 mesons and 2 baryons.

but reasoning above gives $\dim_{\mathbb{C}} \mathcal{M} = N_c^2+1$.

There must be one constraint:

$$\det M = \frac{1}{N_c!}\varepsilon^{i_1\ldots i_{N_c}} \varepsilon_{j_1\ldots j_{N_c}} M_{i_1}^{\ j_1} \ldots M_{i_{N_c}}^{\ j_{N_c}}$$

$$= \frac{1}{N_c!}\varepsilon^{i_1\ldots i_{N_c}} Q_{i_1}^{a_1} \ldots Q_{i_{N_c}}^{a_{N_c}} \varepsilon_{j_1\ldots j_{N_c}} \tilde{Q}_{a_1}^{j_1} \ldots \tilde{Q}_{a_{N_c}}^{j_{N_c}}$$

$$= \frac{1}{N_c!} B \varepsilon^{a_1\ldots a_{N_c}} \tilde{B} \varepsilon_{a_1\ldots a_{N_c}} = B\tilde{B}.$$

$$\det M = B\tilde{B}.$$

* $N_f = N_c+1$ $\qquad \dim_{\mathbb{C}} \mathcal{M} = N_c^2+2N_c+1 = N_f^2$

here we have $B^i = \varepsilon^{i i_1 \ldots i_{N_c}} B_{i_1\ldots i_{N_c}}$,

$\tilde{B}_i = \varepsilon_{i i_1\ldots i_{N_c}} \tilde{B}^{i_1\ldots i_{N_c}}$

$$\to \quad B^i \tilde{B}_j = \det M \, (M^{-1})^i_{\ j}$$

$$B^i M_i^{\ j} = 0 = M_i^{\ j} \tilde{B}_j$$

We can now start discussing quantum (non perturbative) effects.

* Consider first $N_f < N_c$ (actually $N_f < N_c - 1$)

Classically we have a moduli space of vacua parametrized by M_i^j. At each pt we have a pure SYM theory with gauge group $SU(N_c - N_f)$.

It has a β-fn: $\tilde{\beta} = 3(N_c - N_f)$ and a scale $\tilde{\Lambda}$.

We expect, by gaugino condensation, to have an effective superpotential given by:

$$W_{eff} = (N_c - N_f)\, \tilde{\Lambda}^3 \equiv (N_c - N_f)\left(\tilde{\Lambda}^{3(N_c - N_f)}\right)^{\frac{1}{N_c - N_f}}$$

Now the scale $\tilde{\Lambda}$ is related to the scale Λ of the unhiggsed theory by the matching of scales:

$$\tilde{\Lambda}^{3(N_c - N_f)} = \frac{\Lambda^{3N_c - N_f}}{\det M}$$

$\det M$ because it is a singlet of $SU(N_f) \times \widetilde{SU(N_f)} \times U(1)_B$.

What about $U(1)_R$?

$$\frac{1}{\det M} \text{ has charge : } -2N_f \frac{N_f - N_c}{N_f} = 2(N_c - N_f)$$

exactly what one expects of $\tilde{\Lambda}^{3(N_c - N_f)}$

In terms of the $SU(N_c)$ theory, the low energy superpotential is:

$$W_{eff} = (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}}$$

It is a runaway potential:

$$\frac{\partial W_{eff}}{\partial M} = -\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} M^{-1} = 0 \iff M \to \infty .$$

the SUSY vacua are pushed to ∞ $\to$ the theory actually has no vacuum at all!
SUSY breaking, but in a runaway fashion.

~~What if we add mass terms~~

Actually, the form of the superpotential can be fixed by symmetries alone:
invariance under $SU(N_f) \times \widetilde{SU(N_f)}$ $\to$ $\det M$ only can appear.
W_{eff} has R-charge 2 $W_{eff} \propto \left(\frac{1}{\det M}\right)^{\frac{1}{N_c-N_f}}$
(remember Λ has R-charge 0)
by dimensional analysis (W_eff has dimension 3)

$$W_{eff} = \alpha \left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}$$

In this case, α can be fixed by a 1-instanton computation precisely in the case that was excluded before $N_f = N_c - 1$
where $W_{eff} = \frac{\Lambda^{2N_c+1}}{\det M}$ ← this is the 1-instanton e^{-S}.

So the Affleck-Dine-Seiberg W_{eff} is valid for any $N_f < N_c$.

* What if we add a mass term. (massive SQCD)

$$W'_{eff} = \mathrm{tr}\, m_i{}^j M_j{}^i + (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}}$$

It means the quark superfields are massive → we can integrate them out, here at the effective level.

$$\frac{\partial W'_{eff}}{\partial M} = m - \left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} M^{-1} = 0 .$$

$$\det m = \left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{N_f}{N_c-N_f}} \frac{1}{\det M} .$$

$$(\det M)^{N_c} = \left(\Lambda^{3N_c-N_f}\right)^{N_f} \frac{1}{(\det m)^{N_c-N_f}}$$

$$\det M = \frac{\left(\Lambda^{3N_c-N_f}\right)^{\frac{N_f}{N_c}}}{(\det m)^{\frac{N_c-N_f}{N_c}}} \qquad M_i{}^j = (m^{-1})_i{}^j \left(\Lambda^{3N_c-N_f}\right)^{\frac{1}{N_c}} (\det m)^{\frac{1}{N_c}}$$

$$M_i{}^j = (m^{-1})_i{}^j \left(\Lambda^{3N_c-N_f} \det m\right)^{\frac{1}{N_c}}$$

and $$W'_{eff} = (\underset{\substack{\uparrow \\ \mathrm{tr}\, m M \,\propto\, N_f (\quad)^{1/N_c}}}{N_f} + N_c - N_f)\left(\Lambda^{3N_c-N_f} \det m\right)^{1/N_c} = N_c\left(\Lambda^{3N_c-N_f} \det m\right)^{1/N_c}$$

We could have guessed the result: at scales lower than m, we have pure $SU(N_c)$ SYM with scale Λ'

and $W_{eff} = N_c \Lambda'^3$

scale matching: $\Lambda'^{3N_c} = \Lambda^{3N_c-N_f} \det m$ (R-charge: $0 + N_f \cdot \frac{2N_c}{N_f} = 2N_c$)

Actually, from $W'_{eff} = N_c\left(\Lambda^{3N_c-N_f}\det m\right)^{\frac{1}{N_c}}$

we could have obtained W_{ADS} by integrating in.

* What if $\det m = 0$: only some flavors are massive.

Take e.g. $W_{tree} = m\,\cancel{M_{N_f}^{N_f}}\,Q_{N_f}\tilde{Q}^{N_f} = m\,M_{N_f}^{N_f}$

only 1 flavor has a mass.

At the effective level, we should integrate out the effective fields that contain the massive quark superfields:

$M_{N_f}^{N_f}$, $M_{N_f}^{\alpha}$ $M_{\alpha}^{N_f}$ $\alpha = 1 \ldots N_f - 1$.

$$W_{eff} = m\,M_{N_f}^{N_f} + (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}.$$

$$\frac{\partial W_{eff}}{\partial M_{N_f}^{N_f}} = m - \left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}(M^{-1})_{N_f}^{N_f} = 0$$

$$\frac{\partial W_{eff}}{\partial M_{\alpha}^{N_f}} = -\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}(M^{-1})_{N_f}^{\alpha} = 0$$

$$\frac{\partial W_{eff}}{\partial M_{N_f}^{\alpha}} = -\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}(M^{-1})_{\alpha}^{N_f} = 0$$

Thus we have
$$(M^{-1})_i^{\,j} = \cancel{\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{-\frac{1}{N_c-N_f}}}\begin{array}{c} \overbrace{\hphantom{(M^{-1})_\alpha^{\beta}}}^{N_f-1} \hphantom{\;\;\; m(\;)^{-1}} \\ \left(\begin{array}{c|c} (M^{-1})_{\alpha}^{\beta} & 0 \\ \hline 0 & m(\;)^{-1} \end{array}\right) \end{array}$$

Which clearly implies:

$$M_i^{\,j} = \cancel{\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}} \left(\begin{array}{c|c} M'^{\ \beta}_{\alpha} & \\ \hline & m^{-1}() \end{array}\right)$$

$$\Rightarrow \quad M_{N_f}^{\ \alpha} = 0 = M_{\alpha}^{\ N_f} \quad , \; M_{N_f}^{\ N_f} = m^{-1}\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}$$

$$\det M = \det M' \cdot M_{N_f}^{\ N_f} = \det M' \cdot m^{-1}\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}$$

$$(\det M)^{N_c-N_f+1} \cdot m^{N_c-N_f} = (\det M')^{N_c-N_f}\, \Lambda^{3N_c-N_f}$$

$$\det M = \left(\frac{\Lambda^{3N_c-N_f}}{m^{N_c-N_f}} (\det M')^{N_c-N_f}\right)^{\frac{1}{N_c-N_f+1}}$$

$$\frac{\Lambda^{3N_c-N_f}}{\det M} = \left[\frac{(\Lambda^{3N_c-N_f})^{N_c-N_f}\, m^{N_c-N_f}}{(\det M')^{N_c-N_f}}\right]^{\frac{1}{N_c-N_f+1}}$$

$$\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} = \left(\frac{\Lambda^{3N_c-N_f}\, m}{\det M'}\right)^{\frac{1}{N_c-N_f+1}} = m M_{N_f}^{\ N_f}$$

$$\Rightarrow \quad W'_{eff} = (N_c-N_f+1)\left(\frac{\Lambda^{3N_c-N_f}\, m}{\det M'}\right)^{\frac{1}{N_c-N_f+1}}$$

matching of scales: $\Lambda'^{\,3N_c-N_f+1} = \Lambda^{3N_c-N_f}\, m$.

This is W_{eff} for $SU(N_c)$ SQCD with N_f-1 flavors.

W_{ADS} is perfectly consistent with integrating out procedure.

* We now try to extend to $N_f \geq N_c$.

We meet a problem: because of $\frac{1}{N_c - N_f}$ exponent, $N_f = N_c$ is not well defined and for $N_f > N_c$ we have a negative power of Λ in W_{eff}: bad $\Lambda \to 0$ limit. Also, there are more invariants (B) which could appear in W_{eff}.

* Consider first $N_f = N_c$ case.

It is a special case.

	$SU(N_f)$	$SU(\tilde{N}_f)$	$U(1)_B$	$U(1)_R$
$M_i{}^j$	$\bar{N}_f$	N_f	0	0
B	1	1	N_f	0
$\tilde{B}$	1	1	$-N_f$	0

no invariant has R-charge $\rightarrow$ impossible to get W_{eff} of R-charge 2.

$\Rightarrow W_{eff} = 0$ no quantum corrections?

There should be, because by giving a mass to 1 flavor and integrating out, we should obtain W_{ADS} which depends on Λ.

Remember that we have the constraint on moduli space:

$\det M - B\tilde{B} = 0$ it is a singlet of all symmetries, and it is of dimension $2N_c$.

$\rightarrow$ matches the charges and dimension of the one-instanton factor Λ^{2N_c}

We can try to guess a deformation of moduli space and see if it is consistent with integrating out:

$$\det M - B\tilde{B} = \Lambda^{2N_c} \qquad \text{quantum deformed moduli space.}$$

We enforce this by a ~~constr~~ Lagrange multiplier:

$$W_{eff} = X(\det M - B\tilde{B} - \Lambda^{2N_c}) .$$

Add a mass:

$$W'_{eff} = m M_{N_c}{}^{N_c} + X(\det M - B\tilde{B} - \Lambda^{2N_c})$$

$\int$-out all fields with $Q_{N_c}, \tilde{Q}^{N_c}$ in them: $M_{N_c}{}^{N_c}, M_{\alpha}{}^{N_c}, M_{N_c}{}^{\alpha}$
$B, \tilde{B}$.

and impose the constraint.

$$\frac{\partial W'_{eff}}{\partial M_{N_f}{}^{N_f}} = m + X \det M \cdot (M^{-1})_{N_f}{}^{N_f} = 0$$

$$\frac{\partial W'_{eff}}{\partial M_{\alpha}{}^{N_f}} = X \det M \, (M^{-1})_{N_f}{}^{\alpha} = 0 \quad \text{similarly for } M_{N_f}{}^{\alpha}$$

$$\frac{\partial W'_{eff}}{\partial B} = -X\tilde{B} = 0 \qquad \text{similarly for } \tilde{B} .$$

$$\frac{\partial W'_{eff}}{\partial X} = \det M - B\tilde{B} - \Lambda^{2N_c} = 0 .$$

The eqs for $M_{\alpha}{}^{N_f}, M_{N_f}{}^{\alpha}$ imply $M_{\alpha}{}^{N_f} = 0 = M_{N_f}{}^{\alpha}$

$$\rightarrow \quad \det M = \det M' \cdot M_{N_f}{}^{N_f} .$$

The eq. for $M_{N_f}{}^{N_f}$, given that ~~m~~ $m \neq 0$ implies that $X \neq 0$

$(M^{-1})_{N_f}{}^{N_f} = (M_{N_f}{}^{N_f})^{-1}$. ~~It~~ becomes:

The eqs. for B and $\tilde{B}$ imply that $B = 0 = \tilde{B}$ because $X \neq 0$.

Then the constraint reads:

$$\det M' \cdot M_{N_f}{}^{N_f} = \Lambda^{2N_c} \qquad \to M_{N_f}{}^{N_f} = \frac{\Lambda^{2N_c}}{\det M'}$$

and $$W'_{eff} = \frac{\Lambda^{2N_c} m}{\det M'} \quad , \text{where} \quad \Lambda^{2N_c} m = \Lambda'^{2N_c+1}$$

This is exactly W_{ADS} for $N_f = N_c - 1$.

It proves that the deformed moduli space was the right guess.

* A closer look at $\det M - B\tilde{B} = \Lambda^{2N_c}$

Classically, $\det M = B\tilde{B}$ is a singular moduli space.

e.g. $M = 0 = B = \tilde{B}$ is a singular pt, where classically there is gauge enhancement to $SU(N_c)$.

The deformed moduli space is completely smooth:

$$d\left(\det M - B\tilde{B} - \Lambda^{2N_c}\right) = \det M \cdot M^{-1} dM - B\, d\tilde{B} - \tilde{B}\, dB \neq 0$$
$$\text{on } \det M - B\tilde{B} = \Lambda^{2N_c}$$

No pts where extra degrees of freedom could appear.

For large VEVs, Higgs phase. For small VEVs, confining phase more appropriate description.

Chiral symmetry $SU(N_f)\times \widetilde{SU}(N_f)\times U(1)_B\times U(1)_R$

necessarily broken in any pt of moduli space, but different patterns:

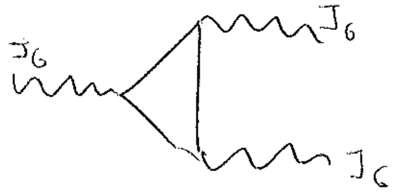
for $M_i^{\,j} = \Lambda^2 \delta_i^{\,j}$, $B=\tilde{B}=0$ $\quad SU(N_f)_{diag}\times U(1)_B\times U(1)_R$ survives

for $M=0$ $\quad B=\tilde{B}=i\Lambda^{2N_c}$ $\quad SU(N_f)\times \widetilde{SU}(N_f)\times U(1)_R$ survives.

* Do we have additional evidence that we are describing the low energy physics with the right degrees of freedom?

→ 't Hooft anomaly matching.

Consider the triangle diagram with only global currents:



currents of global group G (surviving)

~~G = SU(N) ...~~ depends on the pt of moduli space.

The diagram (summed over all fermions of the theory) can be non vanishing, but this has no implications on the theory. Currents are conserved.

However, we could think about (weakly) gauging G. Then, in order for the theory to be consistent, (free of gauge anomalies) we would need to add some fermions in specific reps of G so as to cancel all anomalies.

These fermions are decoupled from anything else, and thus always ~~not~~ give the same contribution to the diagram.

This means that, for G to be anomaly free at all scales, the contribution of the original fermions of the theory must also be the same at all scales. In particular, it must be the same in the UV, where the theory is described by quarks and gaugini, as in the IR, where we have the description in terms of effective fields.

* Let us analyze the $N_f = N_c$ theory at 2 pts of its moduli space.

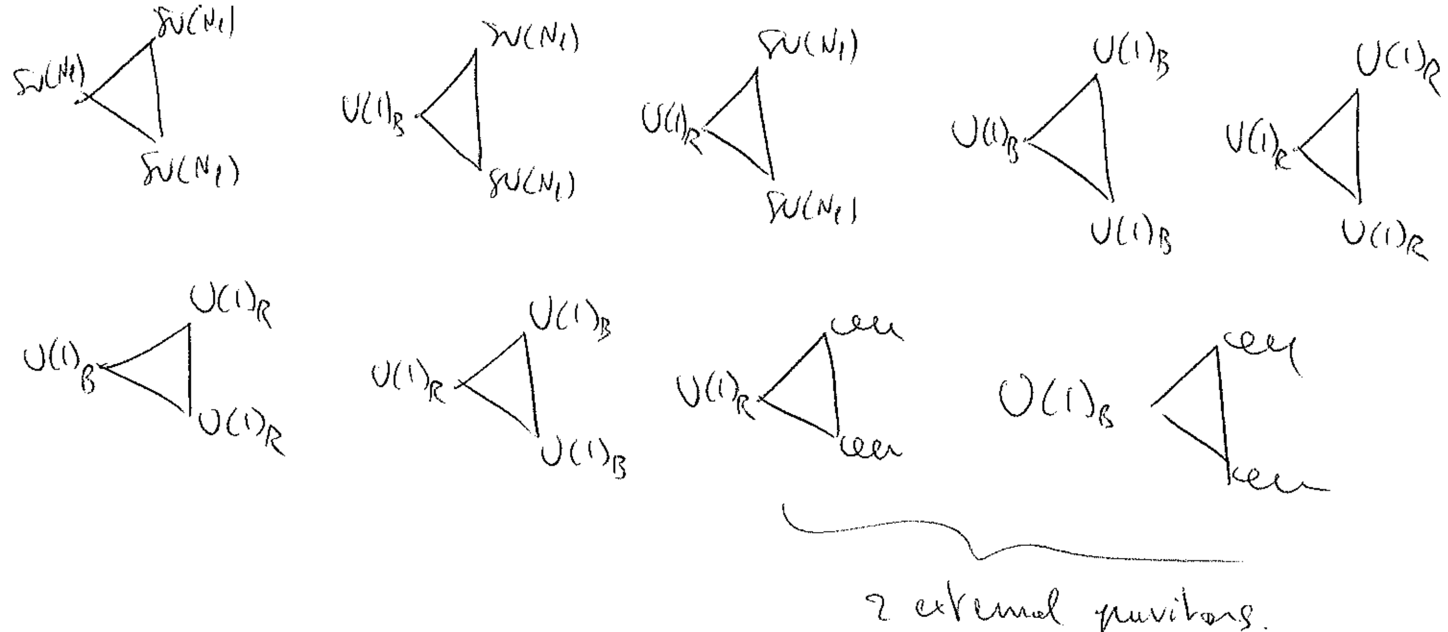
Near $M_i^j = \Lambda^2 \delta_i^j$ $B = \tilde{B} = 0$ the effective fields are the traceless part of M_i^j (the trace is eliminated by the constraint) and the 2 baryons.

The charges under the surviving global group are (the fermions are shown)

	$SU(N_f)_{diag}$	$\times\, U(1)_B$	$\times\, U(1)_R$	
ψ_Q	$\bar{N}_f$	$+1$	-1	(N_c)
$\psi_{\tilde{Q}}$	N_f	-1	-1	(N_c)
λ	1	0	1	(N_c^2-1)
ψ_M	Adj.	0	-1	
ψ_B	1	N_c	-1	
$\tilde{\psi}_B$	1	$-N_c$	-1	

[Recall $R(\psi_\phi) = R(\Phi) - 1$.]

the anomalies we have to check are several:



$SU(N_f)^3$ is zero both in UV and IR because the matter content is already vectorial if $SU(N_f)$ were gauged.

$U(1)_B\, SU(N_f)^2$, $U(1)_B U(1)_R^2$ and $U(1)_B^3$ and $U(1)_B(\text{grav})^2$ are zero both in UV and IR because $\mathrm{tr}\, U(1)_B = 0$ and the ~~other~~ contribution of the other (currents)^2 factorizes.

We are left with 4 non trivial anomalies:

$U(1)_R\, SU(N_f)^2$ UV: $-N_c - N_c = -2N_c$ ~~both~~ $(\psi_q, \tilde{\psi}_q)$ $C_\square = 1 = C_{\bar{\square}}$

IR: $-2N_c$ ✓ $(\psi_M, C_{Adj} = 2N_c)$

$U(1)_R^3$ UV: $N_c^2(-1) + N_c^2(-1) + N_c^2 - 1 = -N_c^2 - 1$

IR: $(N_c^2-1)(-1) + (-1) + (-1) = -N_c^2 - 1$ ✓

$U(1)_R U(1)_B^2$ UV: $N_c^2(-1) + N_c^2(-1) = -2N_c^2$

IR: $N_c^2(-1) + (-N_c)^2(-1) = -2N_c^2$ ✓

$U(1)_R(\text{grav})^2$: UV : $N_c^2(-1) + N_c^2(-1) + N_c^2-1 = -N_c^2-1$

IR : $(N_c^2-1)(-1) + (-1) + (-1) = -N_c^2-1$ ✓

This is a stringent test that is passed.

Let us try the other vacuum: $M=0$ $\quad B=\tilde{B}=i\Lambda^{N_c}$

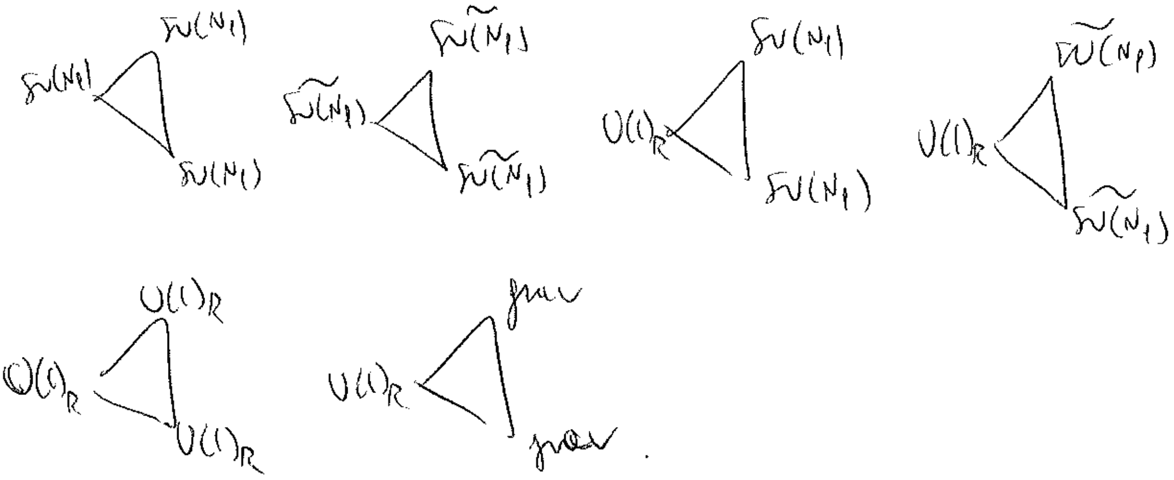
and by the constraint we eliminate $\tilde{B}$.

Under the surviving global group:

$$SU(N_f)\times\widetilde{SU}(N_f)\times U(1)_R$$

ψ_Q	$\bar{N}_f$	1	-1	(N_c)
$\psi_{\tilde{Q}}$	1	N_f	-1	(N_c)
λ	1	1	1	(N_c^2-1)
ψ_M	$\bar{N}_f$	N_f	-1	
ψ_B	1	1	-1	

here we have the following triangles: (recall $\mathrm{tr}\,SU(N_f)=0$)



If we were to gauge, say, $SU(N_f)$, it would be anomalous: both in UV and IR we would have only $\bar{N}_f$, N_f of them.

$\rightarrow$ $SU(N_f)^3$ and $\widetilde{SU}(N_f)^3$ match. ✓

$U(1)_R\, SU(N_f)^2$ $\quad$ UV : $N_c(-1) = -N_c$

$\qquad\qquad\qquad$ IR : $N_f(-1) = -N_c$ ✓

same for $U(1)\, SU(\tilde{N}_f)^2$

$U(1)_R^{\,3}$ $\quad$ UV : $N_c^2(-1) + N_c^2(-1) + N_c^2 - 1 = -N_c^2 - 1$

$\qquad\qquad$ IR : $N_c^2(-1) - 1 = -N_c^2 - 1$ ✓

$U(1)_R(\text{grav})^2$ $\quad$ ~~UV~~: same numerics as above.

Again, perfect matching.

* We now climb up: $N_f = N_c + 1$

	$U(1)_R$
M	$\frac{2}{N_f}$
B	$1 - \frac{1}{N_f}$
$\tilde{B}$	$1 - \frac{1}{N_f}$
$\Lambda^{2N_c - 1}$	~~ARO~~ 0

Recall the invariants : $M_i{}^j$, B^i, $\tilde{B}_j$

which are classically constrained to be:

$$B^i \tilde{B}_j = \det M \cdot (M^{-1})^i{}_j \qquad B^i M_i{}^j = 0 = M_i{}^j \tilde{B}_j .$$

From the R-charges, we see that $\det M$ and $B^i M_i{}^j \tilde{B}_j$ both have R-charge 2. (and dimension $2N_f = 2N_c + 2$)

If we write

$$W_{eff} = \frac{1}{\Lambda^{2N_c - 1}} \left(B^i M_i{}^j \tilde{B}_j - \det M \right)$$

it has R-charge 2 and by taking its extrema reproduces the classical constraints:

$$\frac{\partial}{\partial B^i} W_{eff} = 0 \iff M_i{}^j \tilde{B}_j = 0 \quad \text{similarly for } \tilde{B}_j$$

$$\frac{\partial}{\partial M_i{}^j} W_{eff} = 0 \quad \Leftrightarrow \quad B^i \tilde{B}_j - \det M \cdot (M^{-1})^i_j = 0 .$$

Note that the constraints have ~~dimension~~ non vanishing R-charge, thus cannot possibly have simple corrections as in $N_f = N_c$ case.

Let us get more evidence for this W_{eff} by adding a mass for the last flavor and ∫-it out.

$$W'_{eff} = m M_{N_f}{}^{N_f} + \frac{1}{\Lambda^{2N_c-1}} \left(B^i M_i{}^j \tilde{B}_j - \det M \right)$$

we have to ∫-out $M_{N_f}{}^{N_f}$, $M_\alpha{}^{N_f}$, $M_{N_f}{}^\alpha$ and all B^α, $\tilde{B}_\alpha$ [except B^{N_f}, $\tilde{B}_{N_f}$ which do not contain Q_{N_f}, $\tilde{Q}^{N_f}$]

$$\begin{aligned}
&\partial_{B^\alpha} && \to M_\alpha{}^\beta \tilde{B}_\beta + M_\alpha{}^{N_f} \tilde{B}_{N_f} = 0 \\
&\partial_{\tilde{B}_\beta} && B^\alpha M_\alpha{}^\beta + B^{N_f} M_{N_f}{}^\beta = 0 \\
&\partial M_\alpha{}^{N_f} && B^\alpha \tilde{B}_{N_f} - \det M \cdot (M^{-1})^\alpha_{N_f} = 0 \\
&\partial M_{N_f}{}^\alpha && B^{N_f} \tilde{B}_\alpha - \det M \, (M^{-1})^{N_f}_\alpha = 0
\end{aligned}$$

since $M_\alpha{}^\beta, B^{N_f}, \tilde{B}_{N_f} \neq 0$

$\to B^\alpha = 0 = \tilde{B}_\alpha$

$M_\alpha{}^{N_f} = 0 = M_{N_f}{}^\alpha$

$\det M = \det M' \, M_{N_f}{}^{N_f}$

$B^{N_f} = B'$ $\quad \tilde{B}_{N_f} = \tilde{B}'$

$$\partial M_{N_f}{}^{N_f} \qquad m + \frac{1}{\Lambda^{2N_c-1}} \left(B' \tilde{B}' - \det M' \right) = 0$$

$$\to \det M' - B' \tilde{B}' = \Lambda^{2N_c-1} m \equiv \Lambda'^{2N_c} .$$

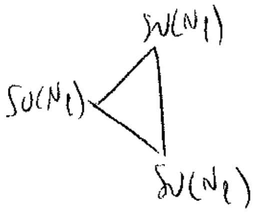
$$W'_{eff} = \frac{M_{N_f}{}^{N_f}}{\Lambda^{2N_c-1}} \left(m \Lambda^{2N_c-1} + B' \tilde{B}' - \det M' \right) = 0 .$$

we exactly get what we had for $N_f = N_c$.

To be convinced that we are handling the right low energy effective degrees of freedom (in particular, all of them are dynamical despite the constraints), we check the 't Hooft anomalies.

At the origin of moduli space, $M_i^j = 0 = B^i = \tilde{B}_j$.
The global symmetry is completely unbroken, yet the theory confines (s-confinement).

	$SU(N_f)$	$\widetilde{SU}(N_f)$	$U(1)_B$	$U(1)_R$	
ψ_q	$\bar{N}_f$	1	1	$-1+\frac{1}{N_f}$	(N_f-1)
$\tilde{\psi}_q$	1	N_f	-1	$-1+\frac{1}{N_f}$	(N_f-1)
λ	1	1	0	1	$N_c^2-1 = N_f^2-2N_f$
ψ_M	$\bar{N}_f$	N_f	0	$-1+\frac{2}{N_f}$	
ψ_B	N_f	1	N_f-1	$-\frac{1}{N_f}$	
$\psi_{\tilde{B}}$	1	$\bar{N}_f$	$-N_f+1$	$-\frac{1}{N_f}$	

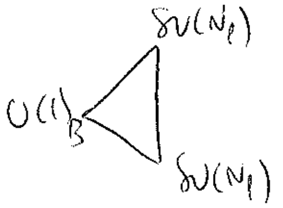


dual theory: UV: N_f-1 $\bar{N}_f$ of $SU(N_f)$

IR: N_f $\bar{N}_f$ and 1 N_f

$\rightarrow$ same "gauge" anomaly.

same goes for $\widetilde{SU}(N_f)^3$.

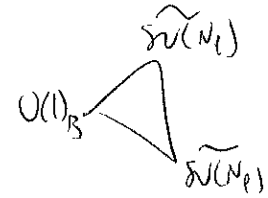


UV: $(N_f-1)\cdot 1 = N_f-1$ only ψ_Q contributes

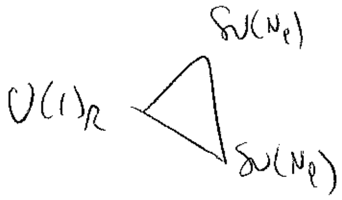
IR: $1\cdot(N_f-1) = N_f-1$ only ψ_B.

✓

similarly



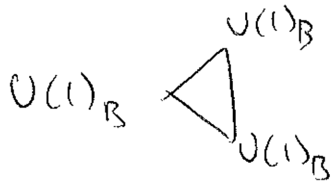
$\rightarrow = 1-N_f$



UV: $(N_f-1)\frac{(-N_f+1)}{N_f} = -\frac{(N_f-1)^2}{N_f}$ (ψ_Q)

IR: $N_f\left(\frac{-N_f+2}{N_f}\right) + \left(-\frac{1}{N_f}\right) = -\frac{1}{N_f}(N_f-1)^2$ ✓

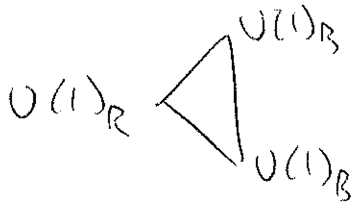
same for $U(1)_R\, \widetilde{SU}(N_f)^2$



UV: $N_f(N_f-1) + (1-1) = 0$

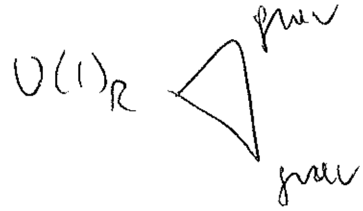
IR: also trivially cancels.

same for $U(1)_B U(1)_R^2$, $U(1)_B(\text{grav})^2$.



UV: $2N_f(N_f-1)\frac{(-N_f+1)}{N_f} = -2(N_f-1)^2$

IR: $2N_f(N_f-1)^2\cdot\left(-\frac{1}{N_f}\right) = -2(N_f-1)^2$ ✓



UV: $2N_f(N_f-1)\left(\frac{-N_f+1}{N_f}\right) + \overset{(N_f-1)^2-1}{\cancel{N_f^2-2N_f}} = -(N_f-1)^2-1$

IR: $N_f^{\cancel{2}}\left(\frac{-N_f+2}{\cancel{N_f}}\right) + 2N_f\left(-\frac{1}{N_f}\right) = -N_f^2+2N_f-2$ ✓

Finally:

$U(1)_R$, $U(1)_R$, $U(1)_R$ (triangle)

$$UV: -2N_f(N_f-1)\frac{1}{N_f^3}(N_f-1)^3 + (N_f-1)^2 - 1 = -2\frac{(N_f-1)^4}{N_f^2} + (N_f-1)^2 - 1$$

$$\left(= -N_f^2 + 6N_f - 12 + \frac{8}{N_f} - \frac{2}{N_f^2}\right.$$

$$IR: -\cancel{N_f^2}\frac{1}{N_f^{\cancel{3}}}(N_f-2)^3 + 2\cancel{N_f}\frac{1}{N_f^{\cancel{3}\,2}} =$$

$$= -N_f^2 + 6N_f - 12 + \frac{8}{N_f} - \frac{2}{N_f^2} \quad \checkmark \quad !!!$$

we thus have the right theory.

* Does this go on for $N_f > N_c + 1$?

Take e.g. $N_f = N_c + 2$ effective fields should be

M_i^j, B^{ij} and $\tilde{B}_{ij}$

at the origin we expect the full global symmetry to be unbroken. The fields charged under the non abelian groups are:

	$SU(N_f)$	$\times\ \widetilde{SU}(N_f)$	
Q	$\bar{\square}$	1	(N_f-2)
$\tilde{Q}$	1	$\square$	(N_f-2)
M	$\bar{\square}$	$\square$	
B	$\bar{\boxminus}$	1	
$\tilde{B}$	1	$\bar{\boxminus}$	

by $\boxminus$ we denote antisymmetric rep of $SU(N)$

Take $SU(N_f)$: in UV as it has an anomaly due to N_f-2 antifundamentals: $\triangle \sim -(N_f-2)$.

In the IR it has N_f antifundamentals and 1 antisymmetric: $\triangle \sim -N_f + (N_f - 4) = -4$

it matches only for $N_f = 6$ (accident).

Already at the stage of the first simple triangle, we see that the anomalies do not match. $\rightarrow$ we should look for a better low energy description ...

* Let us go a little bit ahead and focus on N_f larger, close to $3N_c$.

The one loop beta fn is $\beta_{1\text{-loop}} = 3N_c - N_f$.

and at $N_f = 3N_c$ $\beta = 0$ conformality.

Actually here we have to stray one second from holomorphicity and Wilsonian thinking, and go back to the "real" beta fn.

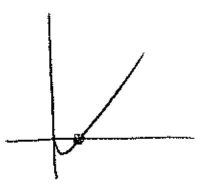
Consider the NSVZ β-fn:

$$\mu \frac{dg}{d\mu} = -\frac{g^3}{16\pi^2} \frac{3N_c - N_f + N_f \gamma(g^2)}{1 - \frac{g^2 N_c}{8\pi^2}}$$

At two loops: $\gamma(g^2) = -\frac{g^2}{8\pi^2}\frac{N_c^2 - 1}{N_c} + O(g^4)$

for N_f close enough to $3N_c$, ~~we see that~~ ($N_f = 3N_c - 1$ e.g.),

$\beta = 0$ implies for $N_c \gg 1$: $1 - 3N_c \frac{g^2}{8\pi^2} N_c = 0 \rightarrow g^2 \sim \frac{8\pi^2}{3N_c^2} \ll 1$



There is an IR fixed pt with conformal symmetry + SUSY $\rightarrow$ superconformal.

Now, superconformal algebra contains a (non anomalous) $U(1)_R$ symmetry.

Let us concentrate on numerator of NSVZ β-fn:
imposing the existence of a conformal fixed pt we get the anomalous dimension of the mesonic operator.
(remember γ related to $\sqrt{Z_Q}$ wave fn. renormalization].

For $\beta=0$ $\quad \gamma = 1 - 3\frac{N_c}{N_f}$.

This implies that M has dimension $\Delta = 2+\gamma = 3\frac{N_f - N_c}{N_f}$

At a superconformal fixed pt, chiral operators must have

$$\Delta = \frac{3}{2}|R| \quad \text{with } R \text{ their } R\text{-charge. (BPS condition)}$$

This implies that M should have R-charge $R = 2\frac{N_f - N_c}{N_f}$.
but this is already the charge of M under the non anomalous $U(1)_R$! $\to$ This already gives a hint that there might be indeed a superconformal fixed pt theory in the IR where M is an effective field.

How much far from $N_f = 3N_c$ can we go?

Note that $\Delta \geq 1$ must hold for unitarity (gauge invariant fields)

$$\Delta(M) \geq 1 \iff 3\frac{N_f - N_c}{N_f} \geq 1 \qquad N_f \geq \frac{3}{2}N_c$$

$\frac{3}{2}N_c < N_f < 3N_c$ is called the conformal window.

Below $N_f < \frac{3}{2} N_c$ presumably the theory is no longer conformal, and $\Delta(M)$ stays 1, as for a free field.

Note also that for $N_f = 3N_c$ $\Delta(M) = 2 = \Delta_d(Q) + \Delta_d(\tilde{Q})$

It looks as ~~though~~ ~~all~~ the quarks still behave as free fields.

Indeed, for $N_f \geq 3N_c$, though UV divergent, the theory is IR free and thus essentially classical at low energies.

* Let us guess the IR superconformal theory.

As in the $N_f = N_c + 1$ theory, we write a tentative trilinear coupling in W_{eff}:

$$W_{eff} = q^i M_i^{\ j} \tilde{q}_j$$

Since $R(M) = \cancel{2 \ldots} \; 2 - 2\frac{N_c}{N_f} \rightarrow R(q) = R(\tilde{q}) = \frac{N_c}{N_f}$.

if $q, \tilde{q}$ were quarks of some dual gauge group, they would have an R charge: $R = \frac{N_f - \tilde{N}_c}{N_f}$.

$$\Rightarrow \quad N_f - \tilde{N}_c = N_c \qquad \underline{\tilde{N}_c = N_f - N_c} \; .$$

We will test the proposal that the dual theory is a gauge theory with gauge group $SU(N_f - N_c)$, N_f pairs of dual quarks $q^i, \tilde{q}_j$ and singlet mesons $M_i^{\ j}$ with $W_{tree} = qM\tilde{q}$.

<u>Seiberg duality</u>: both theories have the same IR fixed pt.

Note also: the R-charge of the baryons of the $SU(N_c)$ theory is

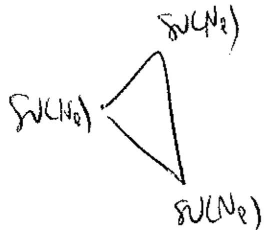
$$R(B)=R(\tilde B)=\frac{N_c(N_f-N_c)}{N_f}\equiv\frac{N_c\tilde N_c}{N_f}=\frac{\tilde N_c(N_f-\tilde N_c)}{N_f}\equiv R(b)=R(\tilde b)$$

that it coincides with the one of the baryons of the dual $SU(N_f-N_c)$ theory. But note under $U(1)_B$ B has charge N_c $\to$ ~~b has charge~~ q has charge $\frac{N_c}{N_f-N_c}$. They are thus identified.

Let us check now the global symmetries and 't Hooft anomaly matching (at the origin where group is biggest).

$$SU(N_f)\times\widetilde{SU}(N_f)\times U(1)_B\times U(1)_R$$

		$SU(N_f)$	$\widetilde{SU}(N_f)$	$U(1)_B$	$U(1)_R$	
electric	ψ_Q	$\overline{N_f}$	1	$+1$	$-\frac{N_c}{N_f}$	(N_c)
	$\psi_{\tilde Q}$	1	N_f	-1	$-\frac{N_c}{N_f}$	(N_c)
	λ	1	1	0	1	(N_c^2-1)
magnetic	ψ_M	$\overline{N_f}$	N_f	0	$1-2\frac{N_c}{N_f}$	(1)
	ψ_q	N_f	1	$\frac{N_c}{N_f-N_c}$	$-1+\frac{N_c}{N_f}$	(N_f-N_c)
	$\psi_{\tilde q}$	1	$\overline{N_f}$	$-\frac{N_c}{N_f-N_c}$	$-1+\frac{N_c}{N_f}$	(N_f-N_c)
	$\tilde\lambda$	1	1	0	1	$((N_f-N_c)^2-1)$

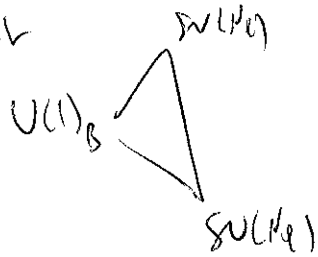


electric theory has anomaly of N_c in the $\bar\Box$ rep.
magnetic has N_f in the $\bar\Box$ and N_f-N_c in $\Box$
$\to$ net anomaly of $N_f-(N_f-N_c)=N_c$ in $\bar\Box$ ✓

Same for $\widetilde{SU}(N_f)^3$.

as usual $\cancel{U(1)_B SU(N_f)^2}$, $\cancel{U(1)_B\widetilde{SU}(N_f)^2}$, $U(1)_B^3$, $U(1)_B U(1)_R^2$, $U(1)_B(\text{grav})^2$ are trivially zero on both sides.

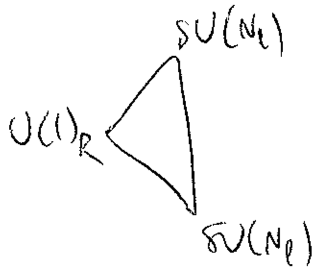
but



electric: N_c

magnetic $(N_f-N_c)\cdot\frac{N_c}{N_f-N_c} = N_c$ ✓

same for $U(1)_B\, \widetilde{SU(N_f)}^2$. ✓

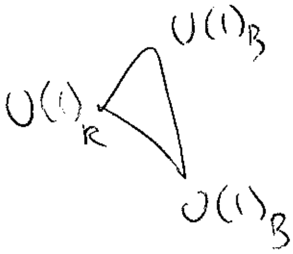


electric: $N_c\cdot\left(-\frac{N_c}{N_f}\right) = -\frac{N_c^2}{N_f}$

magnetic: $N_f\left(1-2\frac{N_c}{N_f}\right) + (N_f-N_c)\left(-1+\frac{N_c}{N_f}\right) =$

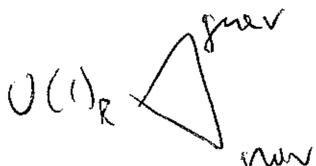
$= \frac{1}{N_f}\left(N_f^2 - 2N_fN_c - (N_f-N_c)^2\right) = -\frac{N_c^2}{N_f}$ ✓

same for $U(1)_R\, \widetilde{SU(N_f)}^2$.



electric: $2N_cN_f\left(-\frac{N_c}{N_f}\right) = -2N_c^2$

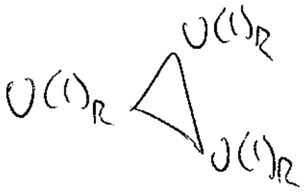
magnetic $2\cancel{(N_f-N_c)}\cancel{N_f}\frac{N_c^2}{\cancel{(N_f-N_c)^2}}\frac{\overset{-1}{\cancel{N_c-N_f}}}{\cancel{N_f}} = -2N_c^2$ ✓



electric: $2N_cN_f\left(-\frac{N_c}{N_f}\right) + N_c^2-1 = -N_c^2-1$

magnetic: $N_f^2\left(1-2\frac{N_c}{N_f}\right) + \cancel{2(N_f-N_c)N_f}\frac{N_c-N_f}{\cancel{N_f}} + (N_f-N_c)^2-1$

$= N_f^2 - 2N_fN_c - (N_f-N_c)^2 - 1 = -N_c^2-1$ ✓



electric: $\cancel{N_f} - 2N_cN_f\frac{N_c^3}{N_f^3} + N_c^2-1 = -2\frac{N_c^4}{N_f^2} + N_c^2 - 1$

magnetic: $N_f^2\left(1-2\frac{N_c}{N_f}\right)^3 + 2(N_f-N_c)N_f\frac{(N_c-N_f)^3}{N_f^3} + (N_f-N_c)^2 - 1 =$

$= \frac{1}{N_f}\left(\cancel{N_f^3} - 6\cancel{N_f^2}N_c + 12\cancel{N_f}\cancel{N_c^2} - \cancel{8N_c^3}\right) - \frac{2}{N_f^2}\left(\cancel{N_f^4} - 4\cancel{N_f^3}N_c + 6\cancel{N_f^2}\cancel{N_c^2} - 4\cancel{N_f}\cancel{N_c^3} + N_c^4\right) + \cancel{N_f^2} - 2\cancel{N_f}N_c + N_c^2 -$

$= N_c^2 - 2\frac{N_c^4}{N_f^2} - 1$ ✓

The anomalies match, so we are more confident that indeed the 2 theories share the same IR dynamics.
The "elementary" degrees of freedom of the magnetic theory are the dual quarks and the singlet mesons (identified with the electric mesons). For instance, the baryons are composite fields in both dual theories.

* ~~What about duality twice~~?
Go back to consider conformal window.
We have noted that the dual theories flow to same IR superconformal pt when $\frac{3}{2}N_c < N_f < 3N_c$.

Note: $N_f < 3N_c \rightarrow 3(N_c - N_f) > -2N_f \rightarrow N_f > \frac{3}{2}\tilde{N}_c$.

$N_f > \frac{3}{2}N_c \rightarrow -\frac{1}{2}N_f > \frac{3}{2}(N_c - N_f) \qquad N_f < 3\tilde{N}_c$

magnetic theory in the same window.

What happens in the electric theory when $N_c + 2 \leq N_f \leq \frac{3}{2}N_c$?
strongly coupled in IR.
We have argued that $\Delta(M) = 1$ there, as for a free theory.
The anomaly matching is still correct. So it sounds like it is still a theory with gauge group $SU(N_f - N_c)$ w/ dual quarks and singlet mesons.

$\rightarrow$ β-fct: $\quad \beta_{1\text{-loop}} = 3\tilde{N}_c - N_f = 3N_f - 3N_c - N_f = 2N_f - 3N_c < 0$

it is UV divergent $\rightarrow$ but IR free.
Conjecture: IR dynamics described by a free magnetic theory.

* What happens when we dualize twice?

electric theory: $SU(N_c)$, N_f $Q_i, \tilde{Q}^i$, $W=0$

magnetic theory: $SU(N_f-N_c)$ N_f, $q^i, \tilde{q}_i$, N_f^2 M_i^j, $W=\frac{1}{\hat{\Lambda}} q^i M_i^j \tilde{q}_j$

? theory: $\tilde{\tilde{N}}_c = N_f-(N_f-N_c)=N_c$

N_f flavors: r_i, $\tilde{r}^i$, N_f^2 N^i_j, and also M_i^j unaffected.

$$W_{tot} = \frac{1}{\hat{\Lambda}} M_i^j N^i_j + \frac{1}{\hat{\Lambda}'} r_i N^i_j \tilde{r}^j$$

Now, M and N appear quadratically → can be integrated out.

$$\frac{\partial W_{tot}}{\partial M_i^j} \to N^i_j = 0$$

$$\frac{\partial W_{tot}}{\partial N^i_j} \to \frac{1}{\hat{\Lambda}} M_i^j + \frac{1}{\hat{\Lambda}'} r_i \tilde{r}^j = 0$$

if we identify $\hat{\Lambda}' = -\hat{\Lambda}$ → $M_i^j = r_i \tilde{r}^j$

$\Rightarrow$ $r_i \equiv Q_i$, $\tilde{r}^i \equiv \tilde{Q}^i$, $W=0$

$\Rightarrow$?-theory is just the original electric theory!

* What is the relation among the scales Λ of electric and $\tilde{\Lambda}$ of magnetic theory?

1-instanton factors: $\Lambda^{3N_c-N_f}$, $\tilde{\Lambda}^{3(N_f-N_c)-N_f} = \tilde{\Lambda}^{-3N_c+2N_f}$

guess is: $\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3(N_f-N_c)-N_f} \propto \hat{\Lambda}^{N_f}$

basically, $\tilde{\Lambda}$ and $\hat{\Lambda}$ determined only in terms of Λ → rescaling freedom.

How to check: add masses to the electric quarks $W_{tree} = m Q \tilde{Q}$

$\to$ IR theory should be $SU(N_c)$ SYM in both cases, with

$W_{eff} = N_c \Lambda_{low}^3$.

- From the electric pt of view: $\Lambda_{low}^{3N_c} = \left(\Lambda^{3N_c - N_f} \det m\right)^{\frac{1}{N_c}}$.

- From the magnetic pt of view: $W = \frac{1}{\hat{\Lambda}} M q \tilde{q} + m M$

M has to have a VEV (as can be obtained by $\langle M \rangle = \frac{\partial W_{eff}}{\partial m}$)

$\to$ it gives a mass to $q, \tilde{q}$. $\to$ we integrate out $q, \tilde{q}$ with mass $\frac{M}{\hat{\Lambda}}$.

$$W_{eff} = (N_f - N_c)\left(\tilde{\Lambda}^{3(N_f-N_c)-N_f} \frac{1}{\hat{\Lambda}^{N_f}} \det M\right)^{\frac{1}{N_f-N_c}} + \mathrm{tr}\, m M.$$

Now we integrate out M:

$$m = M^{-1}\left(\tilde{\Lambda}^{3(N_f-N_c)-N_f} \frac{1}{\hat{\Lambda}^{N_f}} \det M\right)^{\frac{1}{N_f-N_c}}$$

$$\det m = \left(\tilde{\Lambda}^{3(N_f-N_c)-N_f} \frac{1}{\hat{\Lambda}^{N_f}} \det M^{1-\frac{N_f-N_c}{N_f}}\right)^{\frac{N_f}{N_f-N_c}}$$

$$\det M = \left[(\det m)^{\frac{N_f-N_c}{N_f}} \hat{\Lambda}^{N_f} \frac{1}{\tilde{\Lambda}^{3(N_f-N_c)-N_f}}\right]^{\frac{N_f}{N_c}}$$

$$\tilde{\Lambda}^{3(N_f-N_c)-N_f} \frac{1}{\hat{\Lambda}^{N_f}} \det M = (\det m)^{\frac{N_f-N_c}{N_c}} \hat{\Lambda}^{N_f \frac{N_f-N_c}{N_c}} \tilde{\Lambda}^{[3(N_f-N_c)-N_f]\frac{N_c-N_f}{N_c}}$$

$$\to W_{eff} = -N_c\left(\tilde{\Lambda}^{[3(N_f-N_c)-N_f]} \hat{\Lambda}^{N_f} \det m\right)^{\frac{1}{N_c}}$$

$$\Lambda^{3N_c-N_f} = (-1)^{N_c} \tilde{\Lambda}^{-[3(N_f-N_c)-N_f]} \hat{\Lambda}^{N_f}.$$

* Add mass only to one electric quark: $W_{tree} = m\, Q_{N_f} \tilde{Q}^{N_f}$.

on the electric side:

→ $SU(N_c)$ $N_f - 1$ flavors with scale $\Lambda'^{\,3N_c - N_f + 1} = \Lambda^{3N_c - N_f} m$.

on the magnetic side

$$W = m M_{N_f}{}^{N_f} + \frac{1}{\hat{\Lambda}} M_i{}^j q^i \tilde{q}_j$$

integrating out $M_{N_f}{}^{N_f}$ → gives a ~~mass to~~ VEV to $q^{N_f}, \tilde{q}_{N_f}$.

$$\frac{\partial W}{\partial M_{N_f}{}^{N_f}} = m + \frac{1}{\hat{\Lambda}} q^{N_f} \tilde{q}_{N_f} \quad \text{breaks } SU(N_f - N_c) \text{ to } SU(N_f - N_c - 1)$$

but also 1 flavor is eaten → $N_f - 1$ flavors.

scale is given by $\tilde{\Lambda}'^{\,3(N_f - N_c - 1) - N_f + 1} = \dfrac{\tilde{\Lambda}^{3(N_f - N_c) - N_f}}{m \hat{\Lambda}}$.

duality relation:

$$\Lambda'^{\,3N_c - N_f + 1}\, \tilde{\Lambda}'^{\,3(N_f - N_c) - N_f + 1} = \Lambda^{3N_c - N_f} \frac{\tilde{\Lambda}^{3(N_f - N_c) - N_f}}{m} = \hat{\Lambda}^{N_f - 1} \checkmark .$$

Same if we give a VEV to $M_{N_f}{}^{N_f}$ on electric side

→ on magnetic side $W = \frac{1}{\hat{\Lambda}} M_\alpha{}^\beta q^\alpha \tilde{q}_\beta + \frac{1}{\hat{\Lambda}} \langle M_{N_f}{}^{N_f} \rangle q^{N_f} \tilde{q}_{N_f}$

mass to last flavor.

Thus Seiberg duality is thus consistent with integrating out flavors one by one.

Summarize the IR behaviour of theories with N_f flavors as N_f from 0 to $> 3N_c$.

* Start from $N_f = 0$ (SYM)
  - Confinement and chiral symmetry breaking $\mathbb{Z}_{2N} \to \mathbb{Z}_2$.

* $0 < N_f < N_c$ massless flavors:
  - Classical moduli space lifted by quantum corrections (gaugino condensation) → lead to runaway vacua

* $N_f = N_c$ Classical moduli space is deformed at quantum level, removing pts of enhanced gauge symmetry. Confinement/Higgsing + chiral symmetry breaking.

* $N_f = N_c + 1$ Undeformed classical moduli space, confinement (low energy d.o.f.) but <u>no</u> chiral symmetry breaking (at origin of moduli space).

* $N_c + 1 < N_f \leq \frac{3}{2} N_c$: description in the IR by a dual (magnetic) gauge theory which is IR free, with additional singlet mesons and $W = q M \tilde{q}$.

* $\frac{3}{2} N_c < N_f < 3N_c$: IR superconformal fixed pt, which enjoys a dual magnetic description as above (with a kind of strong/weak coupling duality)

* $N_f \geq 3N_c$: since it is no longer asymptotically free, it is on the other hand IR free.

* Supersymmetry breaking: a very brief introduction.

We have said that

$$\begin{array}{ccc} & & V=0 \\ \text{SUSY} & \Leftrightarrow & \Updownarrow \\ & \Leftrightarrow & F=0=D \end{array}$$

Thus an order parameter for SUSY breaking is $V \neq 0$.

Alternatively, $\langle F\rangle \neq 0$ and/or $\langle D\rangle \neq 0$ is a signal for SUSY breaking.

(⚠ We discuss here only spontaneous SUSY breaking and not explicit (though "soft") SUSY breaking, such as adding to the $\mathcal{L} = \ldots + m\lambda^2$.)

What dynamics leads to $\langle F\rangle \neq 0$ or $\langle D\rangle \neq 0$?

Tree level (classical) $\rightarrow$ spontaneous SB

Perturbative: ruled out

Non perturbative: $\rightarrow$ dynamical SB (DSB).

* Take simple example: WZ model with several chiral superfields

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\, \Phi^i \bar\Phi_i + \int d^2\theta\, W(\Phi) + c.c. \qquad \Phi^i = \varphi^i + \theta\psi^i + \theta^2 F^i$$

$$= \text{kinetic terms} + F^i \bar F_i + \partial_i W \cdot F^i + \partial_i\partial_j W \psi^i\psi^j + c.c.$$

after eliminating $F^i = -\partial^i \bar W \qquad \bar F_i = -\partial_i W$

$$\mathcal{L} = (\text{kin}) + \partial_i\partial_j W \psi^i\psi^j + c.c. - V$$

$$V = \partial_i W \partial^i \bar W$$

extremum: $\partial_i V = \partial_i\partial_j W\, \partial^j \bar W$

SUSY breaking: $\partial^j \bar W = -F^j \neq 0 \quad \rightarrow \quad \partial_i\partial_j W F^j = 0$

But $\partial_i \partial_j W$ is the fermionic mass matrix
→ it has one zero eigenvalue.

There is necessarily one massless fermion in a SUSY breaking vacuum. ~~(even if the theory had no massless fermions to begin with (...))~~

This is the SUSY equivalent of the Goldstone theorem.
The massless fermion is the Goldstino. It is obtained by setting $\psi^i \propto f^i \psi_G$. Alternatively, there is another basis of the fermions where $\psi_G \propto f^i \psi^i$.
The supercurrent is indeed $S^\mu_\alpha \sim \sigma^\mu_{\alpha\dot\alpha} \bar\psi_i^{\dot\alpha} f^i \sim \sigma^\mu_{\alpha\dot\alpha} \bar\psi_G^{\dot\alpha}$
Similarly to Goldstone's theorem, broken SUSY implies that the propagator for ψ_G is the one of a massless fermion.

- Tree level breaking:

F-term breaking: simple WZ model

- 1 chiral superfield: $W = \lambda \Sigma$ → $V = |\lambda|^2$ cosmological constant.

- 3 chiral superfields O'Raifeartaigh

$$W = mXY + \lambda Z(X^2 - a^2) \qquad \text{renormalizable}$$

$$\left.\begin{array}{l} \partial_Y W = mX \\ \partial_Z W = \lambda(X^2 - a^2) \end{array}\right\} \text{ cannot both be zero.}$$

Typically has a flat direction of SUSY breaking vacua
→ lifted at one loop.

D-term breaking: Fayet-Iliopoulos

In a gauge theory, if there is a U(1) factor one can write a gauge- and SUSY invariant term as follows:

$$\mathcal{L}_{FI} = \xi \int d^2\theta\, d^2\bar\theta\, \mathrm{tr} V = \xi\, \mathrm{tr} D$$

Example: U(1) gauge theory with Φ_+^{ch} and Φ_-^{ch} of charge ± 1 and a mass term

$$\mathcal{L} = \frac{1}{2} D^2 + \xi D + (\phi_+\bar\phi_+ - \phi_-\bar\phi_-) D + f_+\bar f_+ + f_-\bar f_- + m\phi_+ f_- + m\phi_- f_+ + c.c.$$

$$\rightarrow \quad \bar f_\pm = - m \phi_\mp \qquad D = -(|\phi_+|^2 - |\phi_-|^2 + \xi)$$

$$V = (|\phi_+|^2 - |\phi_-|^2 - \xi)^2 + |m|^2 (|\phi_+|^2 + |\phi_-|^2)$$

$$\partial V = 0 \iff \phi_\pm = 0 \quad \rightarrow V = \xi^2, \ \langle f \rangle = 0 \quad \langle D \rangle = -\xi \neq 0.$$

* Dynamical SUSY breaking

the idea is that because of terms in W generated by non-perturbative effects, one has a kind of F-term SUSY breaking (K non canonical → possibly no massless scalars, but of course Goldstino)

There is one constraint: Witten Index. $\mathrm{tr}(-1)^F$

Only ground states ($V=0$) can be singlets of SUSY.

⇒ if $\mathrm{tr}(-1)^F \neq 0$ there must exist SUSY vacua.

It is the case in SYM, $\mathrm{tr}(-1)^F = N$ for SU(N).

Same for any gauge theory with matter in vectorial (real) rep → by adding mass terms, one goes back to SYM in IR and $\mathrm{tr}(-1)^F \neq 0$.

→ only hope for ~~D~~(stable) DSB are chiral theories.

Matter in reps which are complex : $\rho^* \neq \rho$.

* Examples: $SU(5)$ with matter in $10 + \bar{5} \equiv \boxminus \oplus \bar{\square}$

the contribution to the gauge anomaly of $\boxminus$ is $N-4 \to 1$ for $N=5$ exactly as 1 flavor $\square$.

Peculiarity : no invariant can be written out of 10 and $\bar{5}$
(most non trivial : $5 \notin 10 \otimes 10 \otimes 10$.)

Thus; $W = 0$ ~~at~~ both at tree level and at effective level.
Also, no moduli space (D-flatness)

Global symmetries: 2 $U(1)$s $+ U(1)_R$, one is anomalous.

At low energies : if they are unbroken, there must be 't Hooft anomaly matching → extremely odd. (assuming confinement)
→ they must be broken (at least $U(1)_R$ ~~X~~: by 1-instanton computation one finds $\langle S \rangle \neq 0$)

But if SUSY was unbroken, there would be a non compact non Goldstone boson around → since there is no moduli space at classical level → impossible → SUSY broken.

Clearly $E_{vac} \sim \Lambda^4$ only scale around.

"non calculable".

* Calculable: $SU(3)\times SU(2)$

$$
\begin{array}{lcc}
Q & (3, & 2) \\
M_i & (\bar{3}, & 1) \\
L & (1, & 2)
\end{array}
\qquad W = M_1 Q L
$$

$\dim_{\mathbb{C}} \mathcal{M}$: $14 - 11 = 3$

invariants: $M_1 Q L$, $M_2 Q L$, $Q^2 M_1 M_2$

Classically $W = M_1 Q L \to \begin{array}{l} QL = 0 \\ M_1 Q = 0 \end{array}$ $\Rightarrow$ all cl. $\mathcal{M}$ lifted

Quantum: $W_{eff} = M_1 Q L + \dfrac{\Lambda_3^7}{Q^2 M_1 M_2}$

$$QL - \frac{\Lambda_3^7}{Q^2 M_1^2 M_2} = 0 \qquad M_1 Q L = \frac{\Lambda_3^7}{Q^2 M_1 M_2}, \quad M_2 Q L = 0.$$

$$M_1 Q = 0 \qquad M_1 Q L = 0, \quad Q^2 M_1 M_2 = 0$$

impossible to satisfy all F-terms!

Here: possible to get DSB vacuum for large VEVs, weak coupling.

* Metastable: ISS

SQCD $SU(N_c)$ $N_c < N_f < \frac{3}{2} N_c$ $\qquad W = m Q \tilde{Q}$ $\qquad m \ll \Lambda$

dual magnetic: $W = m^i_j M^j_i + \frac{1}{\hat{\Lambda}} M^j_i q^i_a \tilde{q}^a_j$ $\qquad a = 1 .. N_f - N_c$

K: canonical

$$\frac{\partial W}{\partial M^j_i} = m^i_j + \frac{1}{\hat{\Lambda}} q^i_a \tilde{q}^a_j \neq 0$$

because m rank N_f, $q_a \tilde{q}^a$ rank $N_f - N_c < N_f$.

However SQCD has a vacuum $\to$ only metastable.