

Documentation of the Package “Surface Charges”

Computing conserved charges in gravity using Mathematica

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1 Objective of the package

Computing conserved charges is one of the first thing one usually does when describing a new solution of gravity coupled to matter fields. With the increasing interest in constructing new solutions of gravity for various reasons including field theory/gravity correspondences [1, 2, 3, 4], quantization of three-dimensional gravity [5, 6, 7, 8], etc , it is desirable to being able to define quickly and in a unified way the conserved charges of these solutions. Since the definitions are known, the charges can be computed with an appropriate algorithm in place of paper and pencil. That’s the objective of this package, realized during step by step during the research of the present author.

In Einstein gravity coupled to matter decaying fast at infinity, many methods exist to compute charges in asymptotically flat or asymptotically anti-de Sitter spacetimes. For a sample of the litterature, see e.g. [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. When dealing with gravity with higher curvature corrections, or spacetimes with unconventional asymptotics, a universal method to get the charges is useful. In the approach of [19, 20, 21], conserved charges are defined as a difference of charge between a reference solution, e.g. flat or anti-de Sitter space, and the solution of interest. The only caveat for these methods is that it should exist a path of solutions connecting the reference solution to the one of interest.

If one denotes by ϕ the set of fields of the theory considered, and ξ an (asymptotic) Killing vector of ϕ , the surface charges are integrals of a spacetime $D - 2$ form and one-form in field space $k_\xi[\delta\phi; \phi]$,

$$\mathcal{Q}_\xi[\phi, \bar{\phi}] = \int_{\bar{\phi}}^{\phi} \int_S k_\xi[\delta\phi; \phi] \quad (1)$$

on a $D - 2$ spacetime surface and along a path in field space between the reference solution and the solution of interest. The package SurfaceCharges consists in a set of procedures computing the components $k_\xi^{\mu\nu}[\delta\phi; \phi]$ of this $D - 2$ form,

$$k_\xi[\delta\phi; \phi] = \frac{1}{2(D-2)!} k_\xi^{\mu\nu}[\delta\phi; \phi] \epsilon_{\mu\nu\alpha_1 \dots \alpha_{D-2}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{D-2}} \quad (2)$$

that should then be integrated in spacetime and in field space by the user.

2 Description of the procedures

This package rests on Mathematica v7.0 or higher (compatibility with lower version has not been tested) and the package Riemannian Geometry & Tensor Calculus which can be downloaded at the following RG&TC homepage.

Currently, the theories that can be dealt with are

Einstein gravity:

`ChargesEinstein[...]`

Einstein-Maxwell theory minimally coupled to a scalar:

`ChargesEinsteinMaxwellScalar[...]`

Topologically Massive Gravity:

`ChargesTMG[...]`

Einstein gravity coupled to p-forms:

`ChargesEinstein[...], ChargespForm[...]`

Since the charge form $k_\xi[\delta\phi; \phi]$ is a linear functional of the Lagrangian, one can simply add up the contributions coming from the different pieces of the Lagrangian to get to complete result. When the matter fall-offs quickly at infinity, it does not contribute to the charges. In situations of interest in the adS/CFT correspondence, this is not always the case and the matter should be taken into account in order to obtain the correct finite charges.

The secondary procedures of the package include

- `SeriesWithNTerms[...]`

It is a variation of the function `Series` which computes a series expansion in an asymptotic coordinate up to a given number of terms, in place of up to a given order in the asymptotic coordinate. It is a substitute of `Series` when the metric has components with very different fall-off behaviors.

- `Cotton[...]`

Computes the Cotton tensor in three-dimensional gravity. The commands

`ChargesEinsteinAsympt[...], ChargesEinsteinMaxwellScalarAsympt[...],
ChargesTMGAsympt[...], ChargespFormAsympt[...], CottonAsympt[...]`

are redundant with `ChargesEinstein[...]`, ... with the only difference that `SeriesWithNTerms` is used to expand asymptotically the fields introduced as an input of these commands. They are useful to simplify the computations in the asymptotic context. Note that the metric and the Christoffel symbols should be expanded asymptotically independently.

- `Commutator[...]`, `Liediffd[...]`, `Liediffdd[...]`, `LiediffU[...]`

Computes the commutator of two vector fields, the Lie derivative of a form, a two-form and a vector (the notation follows the one of `RGtensors`, where covariant indices are written as `d`, and contravariant as `U`).

- `GenerateEpsilon[D]`

for defining the global variable `Epsilon` which is the totally anti-symmetric numerically invariant pseudo-tensor in D dimensions.

Help All procedures contain a documentation that can be accessed in Mathematica using the question-mark followed by the name of the procedure, e.g. type

`?ChargesEinstein`

for a description of the syntax associated with that command.

3 How to use the procedures in practice

Here is the sequence of operations required for computing charges illustrated on the toy example of the Schwarzschild black hole:

1. Load the `RG&TC` package followed by the `SurfaceCharge` package.
2. Define a metric and a coordinate system in D dimensions. Define also the matter fields $\Phi_m = (\mathbf{A}, \phi, \dots)$: p -forms, scalar fields and so on if necessary. For example,

```
metric = {{-1 + 2 M/r, 0, 0, 0}, {0, 1/(1 - 2 M/r), 0, 0}, {0, 0,
  r^2, 0}, {0, 0, 0, r^2 sin^2(theta)}}
coord = {t,r,theta,phi}
```

3. Define a symmetry of the theory. For example, a symmetry of gravity coupled to p -form field is a couple (ξ, λ) such that

$$\mathcal{L}_\xi g_{\mu\nu} = 0, \quad \mathcal{L}_\xi \mathbf{A} + d\lambda = 0. \quad (3)$$

If the case of pure gravity, a symmetry is just a Killing vector. In the asymptotic context, these identities only hold asymptotically. In order to compute the mass, define

```
xi = {1,0,0,0}
```

4. Run

```
RGtensors[metric,coord,{0,0}]
```

to generate the Christoffel symbols and the Riemann tensor. In the asymptotic context, let say $r \rightarrow \infty$ it is sufficient and more efficient to run

```
RGtensors[SeriesWithNTerms[metric,N,{r,infinity}],coord,{0,0}]
```

where N is the number of relevant terms to keep in the r expansion (typically 3).

5. Define the surface S of integration of the charges by specifying the two coordinates that remain fixed on the surface. Typically, the sphere at infinity is defined as the constant time and radius surface. In the coordinates $\{t, r, \theta, \phi\}$, the surface will be identified by

```
surface = {1,2}
```

6. Define the perturbed solution by varying all parameters of the solution that are not fixed by the Lagrangian.

```
perturbation = D[metric,M] deltaM
```

7. Run the Charge procedures relevant for the Lagrangian considered: either ChargeEinstein, ChargeEinsteinMaxwellScalar, ChargeTMG, ChargeForm or a combination of them. There is no implementation of the Chern-Simons terms for p -forms, so such terms should be added by hand if necessary. Here,

```
LocalVariationOfTheMass = ChargesEinstein[xi, perturbation, -1, surface]
```

gives the result

```
LocalVariationOfTheMass = sin(theta) deltaM / (4 Pi)
```

8. Integrate the result on the $D - 2$ surface. Note that the factor $\sqrt{-g}$ has been taken care of in the procedure ChargesEinstein and the other “Charges” procedures.

```
VariationOfTheMass = Integrate[LocalVariationOfTheMass,  
                                {theta,0,Pi},{phi,0,2 Pi}]
```

9. If the integrability conditions are obeyed, one can integrate the linearized charge in the phase space between the solution of interest and the background.

$$\text{Mass} = \text{Integrate}[\text{VariationOfTheMass} /. \{\text{deltaM} \rightarrow 1\}, M]$$

to finally get

$$\text{Mass} = M$$

More examples can be found on the webpage.

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