COMPÈRE PHYS-F484

GRAVITATIONAL WAVES

Contents

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[1] Michele Maggiore. *Gravitational Waves. Vol. 1: Theory and Experiments*. Oxford University Press, 2007; Luc Blanchet. Post-newtonian theory for gravitational waves, 2024; and Nathalie Deruelle and Jean-Philippe Uzan. *Relativity in Modern Physics*. Oxford Graduate Texts. Oxford University Press, 8 2018

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Essentials of Gravitational Waves

1.1 A Brief Introduction

The concept of gravitational waves was first proposed by Albert Einstein in his theory of General Relativity in 1916. According to this theory, accelerating objects, or more precisely, accelerating quadrupoles, as we will discuss below, such as merging black holes or neutron stars, produce ripples in the structure of spacetime itself, gravitational waves, propagating outward at the speed of light. However, it took many decades before our humanity had access to the required technology to start detecting them. Quite luckily for us, physicists of the 21st century, we are now living in a world where such observations have not only become possible but are performed daily with prospects of increasing precision in the coming decades, which opens up a new observational window into the cosmos, unprecedented since Galileo's time.

The first observational breakthrough regarding gravitational waves arose in 1974 when Russell Hulse and Joseph Taylor discovered the first indirect evidence of gravitational waves through the study of a binary pulsar system[1,2]. Their observations provided strong support for the existence of gravitational waves, earning them the Nobel Prize in Physics in 1993 "for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation " [3]. They observed that the orbit period of the pulsar is declining: the two bodies are rotating faster and faster about each other in an increasingly tight orbit. The change is very small: it corresponds to a reduction of the orbit period by about 75 millionths of a second per year, but it is nevertheless fully measurable. This change is presumed to occur because the system is emitting energy in the form of gravitational waves. According to astronomical data ranging from 1975 to 2007, the theoretically calculated value from the relativity theory agrees to within about one half of a percent with the observed value, see Figure 1.1.

1.R. A. Hulse and J. H. Taylor. Discovery of a pulsar in a binary system. *Astrophysical Journal*, 195:L51–L53, January 1975

2.J. M. Weisberg and J. H. Taylor. Relativistic binary pulsar b1913+16: Thirty years of observations and analysis, 2004

3.The Nobel Prize in Physics 1993. NobelPrize.org. Nobel Prize Outreach AB 2024. Thu. 9 May 2024. URL <https://www.nobelprize.org/prizes/physics/1993/summary/>

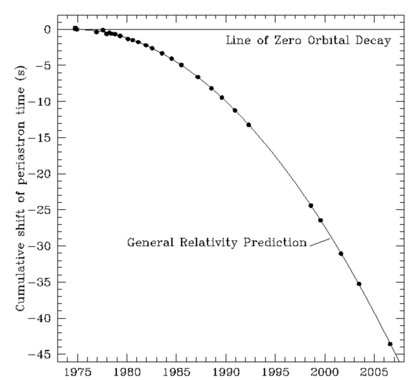


Figure 1.1: Orbital decay of pulsar PSR B1913+16. The data points indicate the observed cumulative shift of the periastron time while the parabola is the prediction of General Relativity: the pulsar emits energy in the form of gravitational waves which changes its periastron.

The quest for direct detection has a long history [4]. The design of the gravitational-wave detector that turned out to win the race for the first detection is based on test masses suspended as pendulum - separated by a long distance. Laser interferometry is used in order to sense the motion of the masses produced as they interact with gravitational waves. Ground-based (i.e. Earth-based) detectors of this type are based on the pioneering work of Moss et al. in the seventies from the Hughes Research Laboratories [1], Rainer Weiss from MIT [5], Drever and colleagues from Glasgow/Caltech between others. These initial designs led to the development of more sophisticated detectors.

[4] For a detailed account, see Matthew Pitkin, Stuart Reid, Sheila Rowan, and Jim Hough. Gravitational Wave Detection by Interferometry (Ground and Space). *Living Rev. Rel.*, 14: 5, 2011. DOI: 10.12942/lrr-2011-5

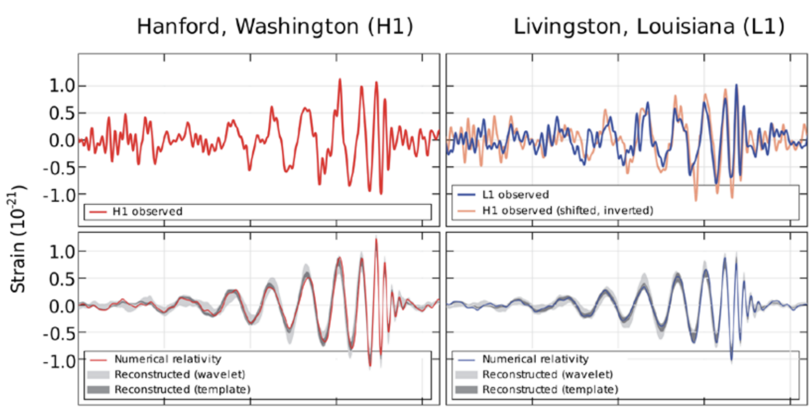
[1] G. E. Moss, L. R. Miller, and R. L. Forward. Photon-noise-limited laser transducer for gravitational antenna. *Appl. Opt.*, 10(11):2495–2498, Nov 1971. DOI: 10.1364/AO.10.002495. URL https://opg.optica.org/ao/abstract.cfm?URI=ao-10-11-2495

[5] See the original article https://dspace.mit.edu/handle/1721.1/146623

In 2002, the Laser Interferometer Gravitational-Wave Observatory (LIGO) became operational, combining two interferometers located in Hanford, Washington, and Livingston, Louisiana. In September 2015, Advanced LIGO, an upgraded version of the detector, began its first observing run at significantly improved sensitivity levels. Still within its engineering phase, even before its official science phase, on September 14, 2015, LIGO made history by directly detecting gravitational waves for the first time.

Example 1.1.1 (Event *GW*150914)**.** *First direct detection of gravitational waves by LIGO/Virgo collaboration.* [2] *According to the best fit values,*

[2] B. P. et Al. Abbott. Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.*, 116:061102, Feb 2016. URL https://link.aps.org/doi/10.1103/PhysRevLett.116.061102



the progenitor masses of the two initial black holes are respectively $M_1 \sim 36M_\odot{}^{+5}_{-4}$ *and* $M_2 \sim 29M_\odot{}^{+4}_{-4}$*, giving rise to a black hole of final mass* $M_{\text{final}} = 62M_\odot{}^{+3}_{-3}$ *and spin* $s = J/M^2 = 0.68 \pm 0.05$*. Therefore, the radiated energy is* $3M_\odot{}^{+0.5}_{-0.5}$*, which is enormous! The luminosity distance is* $d_L = 420^{+150}_{-180}$ *Mpc, which corresponds to a redshift of* $0.09^{+0.03}_{-0.04}$.

The signal *GW*150914, named based on the date of observation, lasted 0.2 seconds and had such an extremely high signal-to-noise ratio that the signal was even visible by eye in the data. The statistical significance of the even is 5.1σ, one order of magnitude higher than the 4σ detection threshold. The best fit template that reproduces the data corresponds to the merger of two black holes, in complete agreement with a key prediction of Einstein's theory. It took 1.3 billion years for the signal to arrive on Earth. This groundbreaking discovery opened a new era in astronomy. In recognition of their contributions, the 2017 Nobel Prize in Physics was awarded to Rainer Weiss, Barry C. Barish, and Kip S. Thorne, "for decisive contributions to the LIGO detector and the observation of gravitational waves".

Since then, LIGO, along with the Virgo interferometer in Italy, has made several more detections of gravitational waves. In 2017, they recorded the first multi-messenger detection of a binary neutron star merger (signal $GW170817$) . These observations have offered unprecedented insights notably into the astronomy of black holes and neutron stars, in cosmology, and on the dynamics of gravity in the strong field regime.

We can enumerate three different types of messengers that give us direct information about the universe:

- Electromagnetic waves;
- Gravitational waves;
- Particles, such as electrons, neutrons, protons and neutrinos.

In this view, gravitational waves provide since 2015 a new means to learn about the universe.

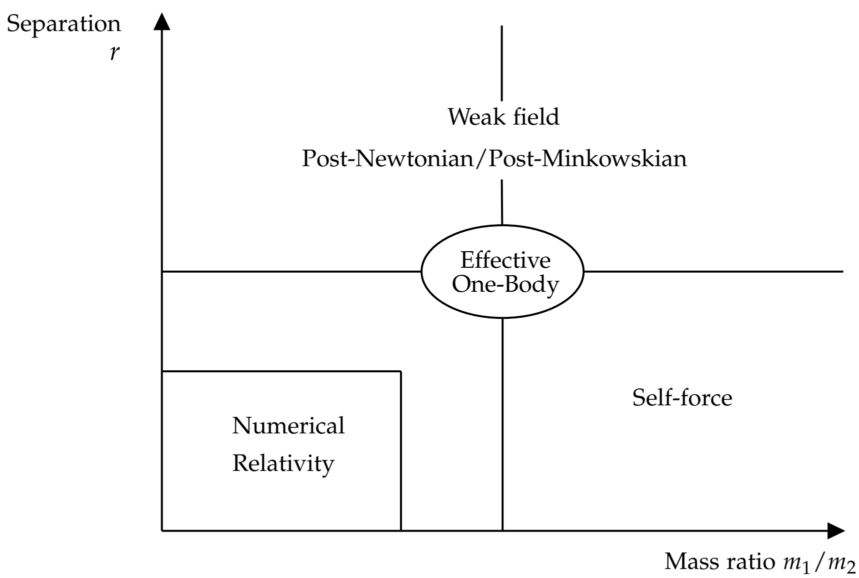
There are four methods used to generate waveforms for binary mergers, which are combined to formulate fast waveform templates, that are in turn confronted to observation. These methods can be depicted on the following figure. The x axis is the mass ratio between the primary and secondary compact object, which ranges from 1 to infinity. The y axis is the separation or distance between the two bodies.

In the large separation limit, gravity is weak away from the bodies, and a post-Minkowskian approximation can be used away from the bodies. At large separation, compact bodies have a relative velocity small compared to the speed of light and a post-Newtonian approximation can be used to describe the gravitation field close to the bodies. The combined use of post-Minkowskian and post-Newtonian expansions is usually denoted as the PN/PM formalism.

In the large mass ratio limit, the secondary object moves along an approximate geodesic during the inspiral around the primary massive object. The corrections to the geodesic motion originate from the finite size effects of the secondary and from the radiation-reaction effects due to the emission of gravitational radiation. The theory that describes such a setting is usually denoted as self-force theory. There is a region of overlap between the large separation and the large mass ratio regime where both the PN/PM and self-force methods apply and can be cross-checked.

In the nearly equal mass ratio regime and at small separation, numerical relativity is applicable. A typical simulation of a binary of mass ratio 10 for around 10 cycles takes several months on a supercluster of supercomputers. The processing time increases with the mass ratio because the separation of scales between the spacetime region close to the secondary as compared with the region close to the primary requires an adaptative mesh with errors that scale with the separation of scales. More asymmetric mass ratios also means that the number of cycles to be computed for relevant frequencies in band for the detector also increases. The processing time increases with the number of cycles to be resolved due to the accumulation of numerical errors during each cycle.

The effective one-body methods are applicable in all ranges of parameters and need to be informed by the various other methods.



1.2 *Post-Newtonian and Post-Minkowskian Theory*

Historically, the weak field approximation of General Relativity was developed at its birth, with the aim to compute deviations from Newtonian theory. Here, we will only develop the necessary tools of the weak field approximation to describe binary coalescences.

In the context of compact binary mergers, one can distinguish at least two length scales: the distance between the compact bodies d and the typical gravitational wavelength $\lambda\!\!\!^{-}_{GW}$ emitted by the system. The typical wavelength associated with the compact binary is $\frac{d}{v/c}$

where v is the typical relative velocity of the compact binary, and it compares to the gravitational wavelength $\lambdabar_{\rm GW}$, as we will show below. For non-relativistic sources, we therefore have $d \ll \lambdabar_{\rm GW}$. At radii $r \ll \lambdabar_{\rm GW}$ post-Newtonian theory is applicable because wave propagation effects can be ignored. For radii $r \gg d$, post-Minkowskian theory is applicable because the gravitational field is approximately described by a perturbation of Minkowski spacetime. Solving for gravitational wave emission from a compact binary system in the weak field regime involves performing these two expansions and matching them in the overlap region $d \ll r \ll \lambdabar_{\rm GW}$. We will consider these two expansions in turn.

Executive summary of General Relativity

Let us first summarize General Relativity. The dynamics of the gravitational field, the metric, is described by the Einstein-Hilbert action

$$\mathcal{S} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g}\, R[g] + \mathcal{S}_M. \tag{1.1}$$

Here $\mathcal{S}_M$ is the matter action, which is minimally coupled to the metric. If one starts with an action around Minkowski spacetime, minimal coupling amounts to apply the rule:

$$\eta_{\mu\nu} \to g_{\mu\nu} \quad ; \quad \partial_\mu \to D_\mu. \tag{1.2}$$

The measure takes the form of $\sqrt{-g} d^4x = \sqrt{-g} c dt d^3x$ where we set $x_0 = ct$. Upon varying the metric, the variation of the matter action allows one to define the matter stress-energy tensor

$$\delta\mathcal{S}_M = \frac{1}{2c} \int d^4x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu}, \tag{1.3}$$

as

$$T^{\mu\nu} = \frac{2c}{\sqrt{-g}} \frac{\delta \mathcal{S}_M}{\delta g_{\mu\nu}}. \tag{1.4}$$

The variation of the Einstein-Hilbert Lagrangian can be written as

$$\delta(\sqrt{-g} R[g]) = -\sqrt{-g} G^{\mu\nu} \delta g_{\mu\nu} + \partial_\alpha \Theta^\alpha[\delta g, g] \tag{1.5}$$

where the last term is a boundary term. The first term is proportional to the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$. Performing the variation of the total action with respect to the metric and requiring invariance of the action under an arbitrary perturbation leads to the Einstein field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.6}$$

General Relativity is invariant under diffeomorphisms which transform the coordinates and the metric as

$$\begin{cases} x^\mu \to x'^\mu(x), \\ g_{\mu\nu} \to g'_{\mu\nu}(x') = \dfrac{\partial x^\rho}{\partial x'^\mu}\dfrac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}\left(x(x')\right). \end{cases} \tag{1.7}$$

A consequence of this invariance are the Bianchi identities, which imply that the stress-energy tensor is covariantly conserved,

$$\nabla^\mu G_{\mu\nu} = 0 \quad \Rightarrow \qquad \nabla^\mu T_{\mu\nu} = 0. \tag{1.8}$$

Propagation around Minkowski

Let us linearize Einstein's theory by considering small metric perturbations around a flat Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1.9}$$

The metric perturbations are small in the sense that the metric components as well as their gradients are small $|h_{\mu\nu}| \ll 1$, $|\partial_\alpha h_{\mu\nu}| \ll 1$ at least in a given set of coordinates.

We will now derive that a wave equation emerges from Einstein's equations in this linearized approximation, and that the solutions can be put in an especially simple form by an appropriate gauge choice. Then, by using standard tools of General Relativity such as the geodesic equation and the equation of geodesic deviation, we will study how these waves interact with a detector, idealized as a set of test masses.

We start by expanding the field equations to linear order in $h_{\mu\nu}$. The resulting theory is called the *linearized theory*. It admits residual gauge transformations

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x), \tag{1.10}$$

which transform the linearized metric as

$$\begin{aligned} h_{\mu\nu}(x) \to h'_{\mu\nu}(x') &= \frac{\partial(x' + \xi(x))^\rho}{\partial x'^\mu}\frac{\partial(x' + \xi(x))^\sigma}{\partial x'^\nu} g_{\rho\sigma}\left(x(x')\right) - \eta_{\mu\nu} \\ &= \partial_\mu\xi_\nu\partial_\nu\xi_\mu + \eta^\alpha\partial_\alpha\eta_{\mu\nu} + h_{\mu\nu}(x). \end{aligned} \tag{1.11}$$

The second equality comes from the fact that

$$\begin{aligned} g_{\rho\sigma}\left(x(x')\right) &= \eta_{\rho\sigma}(x' + \xi) + h_{\rho\sigma}(x) \\ &= \eta_{\rho\sigma}(x') + \xi^\alpha\partial_\alpha\eta_{\rho\sigma} + h_{\rho\sigma}(x). \end{aligned}$$

Consistently with the condition $|h_{\mu\nu}| \ll 1$, the coordinate transformation needs to be small in the sense that $|\partial_\mu\xi_\nu| \ll 1$. The behaviour of the perturbation $h_{\mu\nu}$ under a gauge transformation is thus given by

$$\delta_\xi h_{\mu\nu}(x) = \mathcal{L}_\xi\eta_{\mu\nu}(x), \tag{1.12}$$

where the *Lie derivative* is defined as

$$\mathcal{L}_\xi\eta_{\mu\nu} = \xi^\alpha\partial_\alpha\eta_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{1.13}$$

The isometries of Minkowski spacetime are defined from vector fields ξ_μ that leave the background metric $\eta_{\mu\nu}$ invariant. After a simple algebra, the general solution is given by

Note that in Minkowski coordinates, we have $\partial_\alpha \eta_{\mu\nu} = 0$ because the metric components are constant. However, Minkowski written in spherical coordinates $x^\mu = (ct, r, \theta, \phi)$ will have $\partial_\alpha \eta_{\mu\nu} \neq 0$.

$$\xi_\mu = a_\mu + b_{[\mu\nu]} x^\nu, \tag{1.14}$$

with a_μ, $b_{[\mu\nu]}$ constants that label translations and Lorentz transformations. From Eq. (1.12), such isometries are symmetries of the linearized theory: they leave the linearized metric invariant.

Let us discuss briefly the corresponding finite transformations. Under a finite Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.15}$$

where $\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}$, the metric becomes

Using the metric to raise and lower the indices we have $\Lambda_\alpha{}^\mu \Lambda^\alpha{}_\nu = \delta^\mu_\nu$. Comparing with the definition of the inverse $(\Lambda^{-1})^\mu{}_\alpha \Lambda^\alpha{}_\nu = \delta^\mu_\nu$, we find $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu = \eta_{\nu\alpha} \Lambda^\alpha{}_\beta \eta^{\beta\mu}$.

$$\begin{aligned} g_{\mu\nu}(x) \to g'_{\mu\nu}(x') &= \left(\Lambda^{-1}\right)^\rho{}_\mu \left(\Lambda^{-1}\right)^\sigma{}_\nu g_{\rho\sigma}(x) \\ &= \eta_{\mu\nu} + \left(\Lambda^{-1}\right)^\rho{}_\mu \left(\Lambda^{-1}\right)^\sigma{}_\nu h_{\rho\sigma}(x), \end{aligned} \tag{1.16}$$

such that $h_{\mu\nu}$ is a tensor under Lorentz transformations.

Let us prove that the linear Riemann tensor is gauge invariant in linearized theory. Let us take for convenience the standard Minkowskian coordinates such that $\partial_\alpha \eta_{\beta\gamma} = 0$. We perturb the Christoffel symbols

$$\begin{aligned} \Gamma^\alpha_{\beta\gamma} &= \frac{1}{2} g^{\alpha\lambda} \left(\partial_\beta g_{\gamma\lambda} + \partial_\gamma g_{\beta\lambda} - \partial_\lambda g_{\beta\gamma}\right) \\ &= 0 + \delta\Gamma^\alpha_{\beta\gamma}, \end{aligned} \tag{1.17}$$

such that

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left(\partial_\beta h_{\gamma\lambda} + \partial_\gamma h_{\beta\lambda} - \partial_\lambda h_{\beta\gamma}\right). \tag{1.18}$$

The Riemann tensor is defined as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}. \tag{1.19}$$

Then, by neglecting the terms of order $\mathcal{O}(h^2)$ and lowering and raising indices with the background metric, we find that the Riemann tensor is

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \partial_\rho \Gamma_{\mu|\nu\sigma} - \partial_\sigma \Gamma_{\mu|\nu\rho} \\ &= \frac{1}{2}\left(\partial_\rho \left(\partial_\nu h_{\mu\sigma} + \cancel{\partial_\sigma h_{\mu\nu}} - \partial_\mu h_{\nu\sigma}\right) - \partial_\sigma \left(\partial_\nu h_{\mu\rho} + \cancel{\partial_\rho h_{\mu\nu}} - \partial_\mu h_{\nu\rho}\right)\right) \\ &= \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho}\right). \end{aligned} \tag{1.20}$$

We have now everything in hand to compute the infinitesimal transformation of the Riemann tensor:

$$\begin{aligned}\delta_\xi R_{\mu\nu\rho\sigma} = \frac{1}{2}\big(&\partial_\rho\partial_\nu\partial_\mu\xi_\sigma + \partial_\rho\partial_\nu\partial_\sigma\xi_\mu + \partial_\sigma\partial_\mu\partial_\nu\xi_\rho + \partial_\sigma\partial_\mu\partial_\rho\xi_\nu \\ &-\partial_\rho\partial_\mu\partial_\nu\xi_\sigma - \partial_\rho\partial_\mu\partial_\sigma\xi_\nu - \partial_\sigma\partial_\nu\partial_\mu\xi_\rho - \partial_\sigma\partial_\nu\partial_\rho\xi_\mu\big) \\ = 0,&\end{aligned} \tag{1.21}$$

from which we conclude that the linearized Riemann tensor is invariant under residual gauge transformations.

Linearized Einstein equations

The linearized Einstein equations will simplify upon defining the trace of the perturbation

$$h = \eta^{\mu\nu} h_{\mu\nu}, \tag{1.22}$$

and the trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h. \tag{1.23}$$

The trace of the trace-reversed metric perturbation is just minus the trace of the original metric perturbation,

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h, \tag{1.24}$$

so that we can invert for the metric in terms of the metric-reversed perturbation as

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \tag{1.25}$$

We compute the Ricci tensor and Ricci scalar

$$\begin{cases} R_{\mu\nu} = \eta^{\alpha\beta} R_{\mu\alpha\nu\beta} = \frac{1}{2}\left(2\partial_{(\mu}(\partial\cdot h)_{\nu)} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right), \\ R = \partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h, \end{cases} \tag{1.26}$$

allowing us to compute the linearized Einstein tensor

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ &= \frac{1}{2}\left(2\partial_{(\mu}(\partial\cdot h)_{\nu)} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h\right). \end{aligned} \tag{1.27}$$

Then, we replace $h_{\mu\nu}$ with $\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$, which yields

$$\begin{aligned} G_{\mu\nu} =& \frac{1}{2}\left(2\partial_{(\mu}(\partial\cdot\bar{h})_{\nu)} - \cancel{\partial_\mu\partial_\nu\bar{h}} - \Box\bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\bar{h} + \cancel{\partial_\mu\partial_\nu\bar{h}}\right) \\ &- \frac{1}{2}\eta_{\mu\nu}\left(\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \frac{1}{2}\Box\bar{h} + \Box\bar{h}\right) \\ =& -\frac{1}{2}\left(\Box\bar{h}_{\mu\nu} - \partial_\mu\partial_\alpha\bar{h}^\alpha_\nu - \partial_\nu\partial_\alpha\bar{h}^\alpha_\mu + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta}\right). \end{aligned} \tag{1.28}$$

It is convenient to fix the gauge, at least partially, to the so-called harmonic gauge, which is also called de Donder gauge or Lorentz gauge,

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0.} \tag{1.29}$$

To reach de Donder gauge, we perform the following gauge transformation:

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho\right). \tag{1.30}$$

Therefore, one has

$$\partial^\nu \bar{h}_{\mu\nu} \to \partial^\nu \bar{h}_{\mu\nu} + \Box \xi_\mu. \tag{1.31}$$

If $\partial^\nu \bar{h}_{\mu\nu} \neq 0$, we can solve for ξ_μ up to zero modes of $\Box \xi_\mu = 0$. Let $G(x)$ be a Green's function of $\Box$:

$$\Box G(x-y) = \delta^{(4)}(x-y). \tag{1.32}$$

This yields

$$\xi_\mu = -\int d^4y\, G(x-y)\partial^\nu \bar{h}_{\mu\nu}. \tag{1.33}$$

One can always reach de Donder gauge. In the de Donder gauge, the equations of linearized gravity on a Minkowski background become

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}.} \tag{1.34}$$

This gauge yields four (differential) conditions upon $\bar{h}_{\mu\nu}$. Thus, the initial ten components of $\bar{h}_{\mu\nu}$ reduce to six totally independent components and four dependent components. Note that the stress-tensor obeys $\partial^\mu T_{\mu\nu} = 0$, which can be obtained either by acting on Eq. (1.34) with ∂^μ or by linearizing the contracted Bianchi identity $D^\mu T_{\mu\nu} = 0$.

Transverse-traceless gauge - Propagation

We look at Eq. (1.34) outside of sources, where $T_{\mu\nu} = 0$:

$$\Box \bar{h}_{\mu\nu} = 0, \tag{1.35}$$

with $\Box = -\frac{1}{c^2}\partial_t^2 + \vec{\nabla}^2$. Therefore, gravitational waves propagate at the speed of light. At the moment, the gauge is not totally fixed. We can still use $x^\mu \to x'^\mu = x^\mu - \xi^\mu(x^\alpha)$. The trace-reversed perturbation transforms as

$$\delta_\xi \bar{h}_{\mu\nu} = \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho\right) := \xi_{\mu\nu}, \tag{1.36}$$

and Eq. (1.35) implies that $\xi_{\mu\nu}$ is harmonic:

$$\Box \xi_{\mu\nu} = 0. \tag{1.37}$$

We obtain this equality because the box operator commutes with partial derivatives.

Then, we can shift $\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \xi_{\mu\nu}$, which removes four components of the trace-reversed pertubation. This leaves us with two remaining components, out of ten, which obey a wave equation. Let us fix the gauge further to find those two components.

- We choose ξ^0 such that $\bar{h} = 0$. We have

$$\begin{aligned}\eta^{\mu\nu}(\bar{h}_{\mu\nu} + \xi_{\mu\nu}) &= \bar{h} + 2\partial_\alpha \xi^\alpha - 4\partial_\alpha \xi^\alpha \\ &= \bar{h} - 2\partial_0 \xi^0 - 2\partial_i \xi^i.\end{aligned} \tag{1.38}$$

  Requiring a traceless gauge requires solving the ODE

$$\partial_0 \xi^0 + \partial_i \xi^i = \frac{1}{2}\bar{h}, \tag{1.39}$$

  for $\xi^0(x^0, x^i)$ such that we are left with a harmonic residual gauge parameter $\xi^0(x^i)$. This implies $\bar{h}_{\mu\nu} = h_{\mu\nu}$.

- We choose ξ^i such that $h^{0i} = 0$. To do so, we want

$$h_{0i} + \xi_{0i} = h_{0i} + \partial_0 \xi_i + \partial_i \xi_0 = 0, \tag{1.40}$$

which results in the ODE

$$\partial_0 \xi_i = -h_{0i} - \partial_i \xi_0, \tag{1.41}$$

for $\xi^i(x^0, x^i)$, such that we are left with a harmonic residual gauge parameter $\xi^i(x^j)$.

Moreover, the harmonic gauge imposes two additional conditions

$$\begin{cases} \partial^0 h_{00} + \partial^i h_{0i} = 0 \Leftrightarrow \partial^0 h_{00} = 0 & \text{if } \mu = 0, \\ \partial^j h_{ij} = 0 & \text{if } \mu = i. \end{cases} \tag{1.42}$$

The first condition implies that $h_{00}(\vec{x})$ is a Newtonian potential, but that is irrelevant for propagation. At the end of the day, we are left with the three following conditions:

$$\boxed{h_{0\mu} = 0 \quad ; \quad h_{ii} = 0 \quad ; \quad \partial^i h_{ij} = 0.} \tag{1.43}$$

This is called the *Transverse-Traceless gauge*, or TT gauge. Note that it cannot be chosen inside sources because $\Box \bar{h}_{\mu\nu} \neq 0$. We now decompose the metric into plane waves weighted by a polarization tensor e_{ij}:

$$h_{ij}^{\text{TT}} = e_{ij}(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t}, \tag{1.44}$$

where $k^\mu = \left(\omega/c, \vec{k}\right)$, $\omega/c = |\vec{k}|$, and $\hat{n} = \vec{k}/|\vec{k}|$ gives the direction of propagation. For a plane wave, $\partial^i h_{ij} = 0$ becomes

$$n^i e_{ij} = 0. \tag{1.45}$$

Let us choose, for definiteness $\hat{n} = \hat{e}_z$, the unit vector along the z axis, and a wave propagating in the $+\hat{e}_z$ direction. Imposing the TT gauge gives

$$\boxed{h_{ij}^{\text{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left(\omega\left(t - \frac{z}{c}\right) + \varphi_0\right).} \tag{1.46}$$

This results in the linear solution

$$\begin{aligned} ds^2 = & -c^2 dt^2 + dz^2 + \left(1 + h_+ \cos\left(\omega\left(t - \frac{z}{c}\right) + \varphi_0\right)\right) dx^2 \\ & + \left(1 - h_+ \cos\left(\omega\left(t - \frac{z}{c}\right) + \varphi_0\right)\right) dy^2 \\ & + 2h_\times \cos\left(\omega\left(t - \frac{z}{c}\right) + \varphi_0\right) dxdy. \end{aligned} \tag{1.47}$$

Given a plane wave solution $h_{\mu\nu}(x)$ propagating in the direction $\hat{n}$ outside of sources in the Lorentz gauge, we can go to the TT gauge as follows: first, we introduce the tensor

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j. \tag{1.48}$$

This tensor is symmetric, transverse, and is a projector. Its trace is $P_{ii} = 2$. We can now construct

Transverse : $n^i P_{ij}(\hat{n}) = 0$
Projector : $P_{ik} P_{kj} = P_{ij}$

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}, \tag{1.49}$$

called the Λ-*tensor*. It has the following properties:

(i) projector: $\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$,

(ii) transverse: $\begin{cases} n^i \Lambda_{ij,kl} = 0, \\ n^k \Lambda_{ij,kl} = 0, \end{cases}$

(iii) traceless: $\begin{cases} \Lambda_{ii,kl} = 0 \\ \Lambda_{ij,kk} = 0 \end{cases}$

(iv) symmetric: $\Lambda_{ij,kl} = \Lambda_{kl,ij}$.

More explicitely, the Λ-tensor can be written as

$$\begin{aligned} \Lambda_{ij,kl}(\hat{n}) =& \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\ &+ \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl} + \frac{1}{2} n_i n_j n_k n_l. \end{aligned} \tag{1.50}$$

When working in the de Donder gauge, the two equations

$$\Box h_{ij} = 0 \quad \text{and} \quad \partial^k \bar{h}_{kl} = 0, \tag{1.51}$$

imply that the quantity

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl} h_{kl} \tag{1.52}$$

obeys

$$\Box h_{ij}^{\mathrm{TT}} = 0 \quad \text{and} \quad \partial^k \bar{h}_{kl}^{\mathrm{TT}} = 0. \tag{1.53}$$

By construction, h_{ij}^{TT} is transverse and traceless, but some algebra is required to prove Eq. (1.53). A generic solution to $\Box h_{ij}^{\mathrm{TT}} = 0$ is a superposition of plane waves:

$$h_{ij}^{\mathrm{TT}} = \int \frac{d^3k}{(2\pi)^3} \left(\mathcal{A}_{ij}(\vec{k}) e^{ik\cdot x} + \mathcal{A}_{ij}^*(\vec{k}) e^{-ik\cdot x} \right), \tag{1.54}$$

with $k^\mu = (\omega/c, \vec{k})$, $|\vec{k}| = \omega/c = 2\pi f/c$, $f > 0$ being the frequency, and $\vec{k} = |\vec{k}|\hat{n}$. Therefore, in spherical coordinates, in momentum space, the measure is given by

$$d^3k = |k|^2 d|k| d^2\Omega = \left(\frac{2\pi}{c}\right)^2 f^2 df d^2\Omega, \tag{1.55}$$

$$d^2\Omega = \sin\theta d\theta d\phi. \tag{1.56}$$

In these coordinates,

The metric used is $\eta_{\mu\nu} = diag(-c^2, 1, 1, 1)$.

$$h_{ij}^{\mathrm{TT}} = \frac{1}{c^3}\int_0^\infty df\, f^2 \int d^2\Omega \left(\mathcal{A}_{ij}(f,\hat{n})e^{-2\pi i f\left(t-\frac{\hat{n}\cdot\vec{x}}{c}\right)} + \text{c.c.}\right). \tag{1.57}$$

The TT gauge fixes

The wave of frequency f and direction $\hat{n}$ moves in the $+\hat{n}$ direction.

$$\mathcal{A}^i_i(\vec{k}) = 0, \tag{1.58}$$

$$k^i \mathcal{A}_{ij}(\vec{k}) = 0. \tag{1.59}$$

Note that h_{ij}^{TT} does not reduce to a 2×2 matrix when there are waves in several directions. This is physically the case for stochastic gravitational wave backgrounds. However, for gravitational waves emitted from a single astrophysical source towards a detector on Earth or in space, the direction of propagation of the wave on the celestial sphere, $\hat{n}_0$, is well-defined and we can write

$$\mathcal{A}_{ij}(\vec{k}) = \mathcal{A}_{ij}(f)\delta^{(2)}(\hat{n} - \hat{n}_0). \tag{1.60}$$

We can then span the space transverse to $\hat{n}_0$ by a 2-dimensional space with tensor labels $a, b = 1, 2$: $\hat{n}_0 \cdot \vec{v}_1 = 0 = \hat{n}_0 \cdot \vec{v}_2$, with $v_a = c_1 v_{1,a} + c_2 v_{2,a}$. Then, h_{ij}^{TT} is non-zero only on these transverse directions. Such a transverse tensor is

$$h_{ab}(t,\vec{x}) = \int_0^\infty df \left(\tilde{h}_{ab}(f,\vec{x})e^{-2\pi i f t} + \tilde{h}^*_{ab}(f,\vec{x})e^{2\pi i f t}\right), \tag{1.61}$$

where

$$\begin{aligned}\tilde{h}_{ab}(f,\vec{x}) &= \frac{f^2}{c^3}\int d^2\Omega\, \mathcal{A}_{ab}(f,\hat{n})e^{2\pi i f\frac{\hat{n}\cdot\vec{x}}{c}} \\ &= \frac{f^2}{c^3} A_{ab}(f) e^{2\pi i f \frac{\hat{n}_0 \cdot x}{c}}.\end{aligned} \tag{1.62}$$

In the case of resonant bars and ground-based interferometers, but not in the case of space-based ones, the linear dimensions of the detector are much smaller than the reduced wavelength $\lambda\!\!\!^{-}_{\mathrm{GW}} = \lambda/2\pi$ of the gravitational waves to which they are sensitive:

For example, the LIGO detector has arms of 4 kms long while the maximal sensitivity at around 100 Hz, which corresponds to a wavelength of a million kms.

$$L_{\text{detector}} \ll \lambda\!\!\!^{-}_{\mathrm{GW}}. \tag{1.63}$$

If we choose the origin of the coordinate system within the detector, we have

$$e^{\frac{2\pi i f \hat{n}_0\cdot\vec{x}}{c}} = e^{i\frac{\hat{n}_0\cdot\vec{x}}{\lambda\!\!\!^{-}_{\mathrm{GW}}}} = e^{i\frac{\hat{n}_0\cdot\hat{x}|\vec{x}|}{\lambda\!\!\!^{-}_{\mathrm{GW}}}} \simeq 1, \tag{1.64}$$

for all gravitational waves at the detector location $\vec{x}$ which have $|\vec{x}| \leq L_{\text{detector}}$. We can therefore ignore the $\vec{x}$ dependency and we have, for a single source:

$$h_{ab}(t) = \int_0^\infty df \left(\tilde{h}_{ab}(f)e^{-2\pi ift} + \tilde{h}^*_{ab}(f)e^{2\pi ift}\right). \tag{1.65}$$

However, we have to be careful. We need to keep the $\vec{x}$ dependency when we compare the gravitational wave signal at two different detectors (*e.g.* LIGO Handford and Livingstone), or when we need spatial derivatives of $h_{ab}(t, \vec{x})$ (*e.g.* to compute the stress-energy tensor).

The trace-free and symmetry conditions imply

$$\tilde{h}_{ab}(f) = \begin{pmatrix} \tilde{h}_+(f) & \tilde{h}_\times(f) \\ \tilde{h}_\times(f) & -\tilde{h}_+(f) \end{pmatrix}_{ab}. \tag{1.66}$$

The + and $\times$ polarizations are defined with respect to the given choice of axis in the transverse plane.

The Lorentz transformations that leave the propagation direction $\hat{n}$ invariant are rotations around $\hat{n}$ and boosts in the $\hat{n}$ direction. Under these operations, h_+ and $h_\times$ will transform and mix between themselves. We saw that in the linearized theory, under a Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, the tensor $h_{\mu\nu}$ transforms as

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x). \tag{1.67}$$

Choosing $\hat{n} = \hat{z}$, a rotation around the z axis and a boost along z are written as

$$\Lambda_\mu{}^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_\mu^{\ \nu}, \quad \text{a rotation of angle } \psi. \tag{1.68}$$

$$\Lambda_\mu{}^\nu = \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}_\mu^{\ \nu}, \quad \text{a boost of rapidity } \eta. \tag{1.69}$$

The rapidity is expressed in terms of the velocity as $\eta = \text{arctanh}\frac{v}{c}$.

In the TT gauge, we remind ourselves that h^{TT}_{ij} has no 0 component and is a superposition of plane waves $\mathcal{A}_{ij}(\vec{k})e^{ik\cdot x}$ that are transverse, meaning $h^{\text{TT}}_{iz} = 0$. Since these Lorentz transformations do not affect the direction of the transverse plane, the tensor indices a, b are unambiguous after the Lorentz transformation and we have

Note that $k \cdot x = k' \cdot x'$.

$$\left(h^{\text{TT}}_{ab}\right)'(x') = \begin{pmatrix} h'_+ & h'_\times \\ h'_\times & -h'_+ \end{pmatrix} e^{ik\cdot x}, \tag{1.70}$$

where

$$\begin{pmatrix} h'_+ & h'_\times \\ h'_\times & -h'_+ \end{pmatrix}_{ab} = \Lambda_a{}^c \Lambda_b{}^d \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}_{cd}, \quad a,b,c,d = 1,2. \tag{1.71}$$

For a rotation, the a, b indices span the second and third rows of $\Lambda_\mu{}^\nu$ associated with the rotation. Therefore, if we look at a rotation, we have

$$\begin{aligned}
\begin{pmatrix} h'_+ & h'_\times \\ h'_\times & -h'_+ \end{pmatrix} &= \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \\
&= \begin{pmatrix} h_+\cos\psi - h_\times \sin\psi & h_\times \cos\psi + h_+ \sin\psi \\ h_+ \sin\psi + h_\times \cos\psi & h_\times \sin\psi - h_+ \cos\psi \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \\
&= \begin{pmatrix} h_+\cos(2\psi) - h_\times \sin(2\psi) & h_+ \sin(2\psi) + h_\times \cos(2\psi) \\ h_+ \sin(2\psi) + h_\times \cos(2\psi) & h_\times \sin(2\psi) - h_+ \cos(2\psi) \end{pmatrix},
\end{aligned} \tag{1.72}$$

which leaves us with the following transformations, while considering rotations:

$$\begin{cases} h'_+ &= h_+ \cos(2\psi) - h_\times \sin(2\psi), \\ h'_\times &= h_+ \sin(2\psi) + h_\times \cos(2\psi). \end{cases} \tag{1.73}$$

Under boosts, the matrix $\Lambda_a{}^c$, *i.e.* the 2×2 submatrix made by the second and the third rows, is the identity matrix, which trivially implies that

$$\begin{cases} h'_+ &= h_+, \\ h'_\times &= h_\times. \end{cases} \tag{1.74}$$

The gravitational wave amplitudes h_+ and $h_\times$ are thus invariant under boosts along e_z. Under a rotation along the z-axis,

$$h_\times \pm i h_+ \to e^{\mp 2i\psi} \left(h_\times \pm i h_+ \right). \tag{1.75}$$

This means that the massless graviton has helicity ± 2 with helicity eigenstates $(h_\times \mp i h_+)$. Helicities always come in pairs in a Lorentz-invariant quantum field theory, such as the quantization of linearized Einstein gravity. This is a consequence of CPT symmetry.

1.3 *Interaction with Test Masses, Freely-falling Frame, TT and Detector Frames*

Review on Geodesics

Geodesic equation

One considers a curve described by coordinates $x^\mu_\gamma(\lambda)$, where γ refers to the curve and λ its parametrization. The spacetime interval ds between two points separated by a parametrized interval $d\lambda$ is

$$ds^2\Big|_\gamma = g_{\mu\nu}\big|_\gamma\, dx^\mu\big|_\gamma\, dx^\nu\big|_\gamma = g_{\mu\nu}\big|_\gamma \frac{dx^\mu_\gamma}{d\lambda}\frac{dx^\nu_\gamma}{d\lambda}d\lambda^2. \tag{1.76}$$

The four-velocity is defined as the tangent vector along the curve:

$$u^\mu = \frac{dx^\mu_\gamma}{d\lambda}. \tag{1.77}$$

For a timelike curve, the proper time τ is defined as

$$c^2 d\tau^2 = -\, ds^2\Big|_\gamma = -\, g_{\mu\nu}\big|_\gamma\, u^\mu u^\nu d\lambda^2. \tag{1.78}$$

Using as parameter the proper time, $\lambda = \tau$, the four-velocity is normalized as

$$g_{\mu\nu}\big|_\gamma\, u^\mu u^\nu = -c^2. \tag{1.79}$$

The classical trajectory of a test particle of mass m is obtained by extremizing the action

$$\begin{aligned}\mathcal{S} &= -m\int_{\tau_i}^{\tau_f} d\tau \\ &= -m\int_{\tau_i}^{\tau_f}\sqrt{-\, g_{\mu\nu}\big|_\gamma\, u^\mu u^\nu}\,d\lambda \\ &= -m\int d\lambda\,\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,\delta^{(4)}(x - x_\gamma(\lambda)).\end{aligned} \tag{1.80}$$

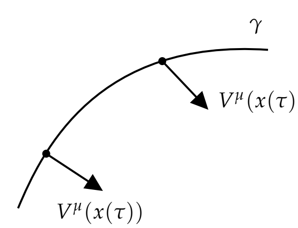
This yields the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{1.81}$$

Proof. See GR class. □

Parallel transport

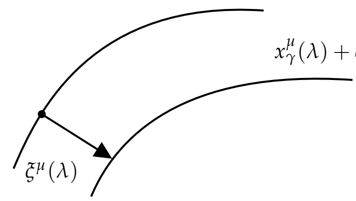
Consider a timelike geodesic $x^\mu_\gamma(\tau)$ and a vector $V^\mu\,(x(\tau))$ defined on the geodesic γ. We define the covariant derivative along the curve $x^\mu(t)$ as



$$\frac{DV^\mu}{D\tau} = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}V^\nu\frac{dx^\rho}{d\tau}. \tag{1.82}$$

The covariant derivative $\frac{DV^\mu}{D\tau}$ transforms as a vector under coordinate transformations. (The proof is left to the reader.) Parallel transport of the vector V^μ along the curve is defined from the condition $\frac{DV^\mu}{D\tau} = 0$).

Geodesic deviation



Consider two geodesics, each one with proper time τ, that are close by. The geodesic equation implies that

$$\frac{d^2(x^\mu+\xi^\mu)}{d\tau^2}+\Gamma^\mu_{\nu\rho}(x^\mu+\xi^\mu)\frac{d(x^\nu+\xi^\nu)}{d\tau}\frac{d(x^\rho+\xi^\rho)}{d\tau}=0. \tag{1.83}$$

If $|\xi^\mu|$ is negligible with respect to the scale of the variation of the gravitational field, we can expand at finite order in $\xi^\mu(\tau)$ such that

$$\frac{d^2\xi^\mu}{d\tau^2}+2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau}+\xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}=0. \tag{1.84}$$

This is the equation of geodesic deviation. We can rewrite it in a more elegant tensorial form as

$$\frac{D^2\xi^\mu}{D\tau^2}=-\xi^\sigma R^\mu{}_{\nu\sigma\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}. \tag{1.85}$$

Proof. By definition,

$$\begin{aligned}\frac{D^2\xi^\mu}{D\tau^2}&=\frac{D}{D\tau}\left(\frac{d\xi^\mu}{d\tau}+\Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau}\right)\\&=\frac{d^2\xi^\mu}{d\tau^2}+\frac{dx^\lambda}{d\tau}\partial_\lambda\Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau}+\Gamma-\text{terms}.\end{aligned} \tag{1.86}$$

As usual, we can work in a frame where the Christoffel symbols (but not their derivatives!) vanish at a point. We then use Eq. (1.84) to write this equation as

$$\begin{aligned}\frac{D^2\xi^\mu}{D\tau^2}&=-\xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}+\frac{dx^\lambda}{d\tau}\partial_\lambda\Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau}+\Gamma-\text{terms}\\&=\xi^\sigma\left(\partial_\nu\Gamma^\mu_{\sigma\rho}-\partial_\sigma\Gamma^\mu_{\nu\rho}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}+\Gamma-\text{terms}\\&=-\xi^\sigma R^\mu{}_{\nu\sigma\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}.\end{aligned} \tag{1.87}$$

□

Interaction of Gravitational Waves with Test Masses

A detector can be idealized as a set of test masses which are described by a reference frame, called the *detector reference frame*. Physical results are invariant under a choice of frame. On the one hand, observations are performed in the detector frame. On the other hand, gravitational waves are most simply described in TT gauge (outside of sources). This gauge corresponds to a specific reference frame, the *TT frame*, which itself is associated with a specific observer. We need to establish a dictionary between these two frames, in order to understand how to describe the gravitational wave observables in terms of the mathematical tools that have been developed in TT gauge.

Definition 1.3.1 (Local inertial frame). *It is always possible to choose coordinates such that the Christoffel symbols vanish at one point. The resulting coordinate system around that point is called a local inertial frame. Moreover, it is always possible to choose coordinates such that on an entire timelike geodesic, all Christoffel symbols vanish.*

Proof. Indeed, under a change of coordinates, we have

$$\Gamma'^{\alpha'}_{\beta'\gamma'} = \frac{\partial x'^{\alpha'}}{\partial x^{\alpha}} \Gamma^{\alpha}_{\beta\gamma} \frac{\partial x^{\beta}}{\partial x'^{\beta'}} \frac{\partial x^{\gamma}}{\partial x'^{\gamma'}} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\beta'} \partial x'^{\gamma'}} \frac{\partial x'^{\alpha'}}{\partial x^{\alpha}}, \tag{1.88}$$

and we can use the inhomogenous terms to set $\Gamma'^{\alpha'}_{\beta'\gamma'} = 0$. For a proof of the second statement we refer to the book of Luther Eisenhart *"Riemannian geometry"*[3]. □

[3] Luther Pfahler Eisenhart. *Riemannian geometry*, volume 51. Princeton university press, 1997

In a local inertial frame of a given geodesic, the geodesic equation becomes around each point P of that geodesic,

$$\left. \frac{d^2 x^{\mu}}{d\tau^2} \right|_P = 0. \tag{1.89}$$

Equivalently, this is the statement that a test mass located at P is freely falling. This gives a realization of the equivalence principle, as a consequence of the modelling of the gravitational field by a metric described by Riemmanian geometry.

An explicit construction of the corresponding system of coordinates goes as follows[4]. At a point P, we choose a basis of four orthonormal four-vectors $e_{\underline{\alpha}}$, $\underline{\alpha} = 0, \dots, 3$ labelling the four-vectors (this is not a spacetime index). We choose them orthogonal to each other with respect to the flat space metric $\eta_{\underline{\mu}\,\underline{\nu}}$. Thus, we have that

[4] James B Hartle. Gravity: an Introduction to Einstein's General Relativity, 2003

The space metric is defined in the tangent space of the manifold at point P, where the 4-vectors $e_{\underline{\alpha}}$ belong.

At point P, we want to build coordinates such that $g_{\mu\nu}\big|_P = \eta_{\mu\nu}$.

$$g_{\mu\nu}\, e^{\mu}_{\ \underline{\alpha}} e^{\nu}_{\ \underline{\beta}} = \eta_{\underline{\alpha}\underline{\beta}}. \tag{1.90}$$

We need to ensure $\Gamma^{\alpha}_{\beta\gamma}\Big|_P = 0$. For that purpose, we consider geodesics that start at P in the direction of a unit four-vector n^{α}. We parametrize the geodesic using the proper distance for a spacelike vector, and proper time for a timelike one. Let Q be a point reached from P after a proper geodesic distance s and let $\left(n^{\underline{0}}, n^{\underline{1}}, n^{\underline{2}}, n^{\underline{3}}\right)$ be the components of n^{α} in the basis $\{e_{\underline{\alpha}}\}$:

$$n^{\underline{\alpha}} = e^{\mu}_{\ \underline{\beta}}\, \eta_{\mu\nu}\, n^{\nu} \eta^{\underline{\alpha}\,\underline{\beta}}. \tag{1.91}$$

We fill all spacetime with such geodesics (null geodesics are obtained as a limit). In a small region of spacetime, geodesics do not intersect (in large regions, we could have gravitational lensing). Thus, each point in a small region is reached by a single geodesic. Coordinates are assigned unambiguously to all points of a sufficiently small spacetime region around P, after a choice of Lorentz frame at P. This

For example, if $n^{\alpha} = e^{\alpha}_{\ \underline{3}}$, by orthogonality, we have $n^{\underline{\alpha}} = (0, 0, 0, 1)$. Then, we assign Q to have coordinates $x^{\alpha}_Q = s n^{\underline{\alpha}}$.

coordinate system is known as *Riemann normal coordinates*. We have $g_{\mu\nu}(P) = \eta_{\mu\nu}$ by construction. In order to show $\Gamma^{\mu}_{\nu\rho}(P) = 0$, we consider the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\nu\rho}(x)\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{1.92}$$

Since the coordinates are linear in proper time by construction, $\dfrac{d^2x^\mu}{d\tau^2} = 0$ at P, while $\dfrac{dx^\mu}{d\tau} = n^\mu$. Then, (1.92) becomes

$$\Gamma^{\mu}_{\nu\rho}\, n^\nu n^\rho \Big|_P = 0. \tag{1.93}$$

Since this holds for all n^μ, we conclude that $\Gamma^{\mu}_{\nu\rho}(P) = 0$. Therefore, the Riemann normal coordinates provide an explicit example of a local inertial frame at a point P.

However, we can do much better than this: building a reference frame where a test mass is in free fall along a timelike geodesic γ. As before, one constructs a Riemann normal coordinate frame in $\underline{3}$ dimensions, orthogonal to the worldline tangent vector, which is chosen as $e^{\mu}_{\underline{0}} = \dfrac{dx^{\mu}_{\gamma}(\tau)}{d\tau}$. At each proper time τ, there is a choice of Lorentz frame to be made. One fixes a choice at proper time $\tau = 0$. The frame is chosen at any τ uniquely by requiring that the frame $e^{\mu}_{\underline{i}}$ is parallely transported along the geodesic γ. We are left with

$$\frac{De^{\mu}_{\underline{i}}}{D\tau} = 0 \quad \Leftrightarrow \quad \frac{de^{\mu}_{\underline{i}}}{d\tau} + \Gamma^{\mu}_{\alpha\beta} e^{\alpha}_{\underline{i}} \frac{dx^{\beta}_{\gamma}(\tau)}{d\tau} = 0. \tag{1.94}$$

Sine $e^{\mu}_{\underline{i}}$ is a vector, $\frac{De^{\mu}_{\underline{i}}}{D\tau}$ is also a vector.

Physically, one can place three gyroscopes along the direction $e^{\mu}_{\underline{i}}$. The gyroscopes will be parallely transported if they are not constrained. These coordinates are called *Fermi normal coordinates*.

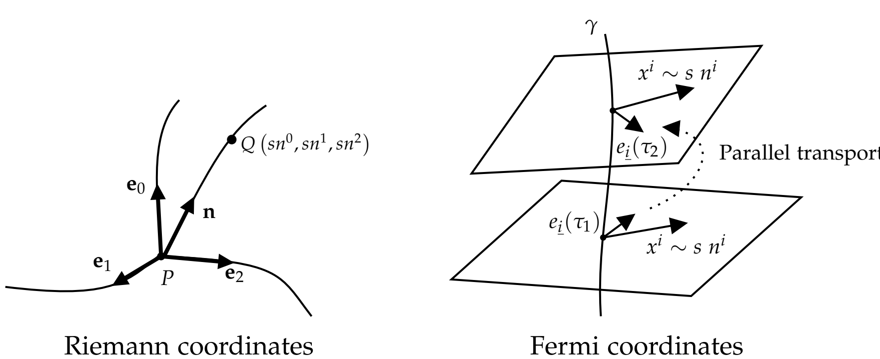


Figure 1.2: Riemann and Fermi coordinate frames. The third space dimension is not depicted: this is the visulization in $2+1$ dimensions.

Such a Fermi frame is realized in practice by drag-free satellites, in which an experimental apparatus is freely-floating inside a satellite that screens it from external disturbances (*e.g.* solar wind, micrometeorites, etc). The satellite locates the experimental apparatus precisely and adjusts its position using thrusters to remain centered about it. This is the core design of the space-based LISA interferometer to be launched around 2036.

TT frame

The coordinate frame in which the metric is in the TT gauge is the *TT frame*. Consider a test mass initially at rest at $\tau = 0$. The geodesic equation is

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = -\,\Gamma^i_{\nu\rho}(x)\left.\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right|_{\tau=0} = -\,\Gamma^i_{00}\left.\left(\frac{dx^0}{d\tau}\right)^2\right|_{\tau=0}, \tag{1.95}$$

where we used the fact that masses are initially at rest: $\left.\frac{dx^i}{d\tau}\right|_{\tau=0} = 0$. For a linear gravitationnal field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, expanding to first order in $h_{\mu\nu}$, the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ become

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}\right), \tag{1.96}$$

and therefore,

$$\Gamma^i_{00} = \frac{1}{2}\left(2\partial_0 h_{0i} - \partial_i h_{00}\right). \tag{1.97}$$

However, in the TT gauge, this quantity vanishes because $h_{0\mu} = 0$, which implies that there is no acceleration:

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = 0. \tag{1.98}$$

If at time $\tau = 0$, $\frac{dx^i}{d\tau}$ is zero, it will remain zero at all times. This shows that in TT frames, particles, which were at rest before the wave passed through, will remain at rest after its passage. The coordinates stretch themselves so that the position of the free test masses does not change. A physical implementation of coordinates can be obtained using the free test masses themselves to mark the coordinates. For example, one could have four test masses defining the origin and three axis, as in Fig. 1.3.

Strictly speakink, this is true at linear order, but $|h| \sim 10^{-2}$, meaning that higher order terms are negligible.

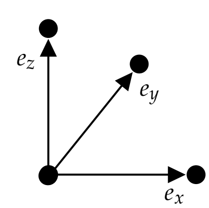


Figure 1.3: Four test masses

What about time? In the TT gauge, have seen that $h_{0\mu} = 0$. The proper time along a timelike trajectory $x^\mu(\tau)$ is obtained from

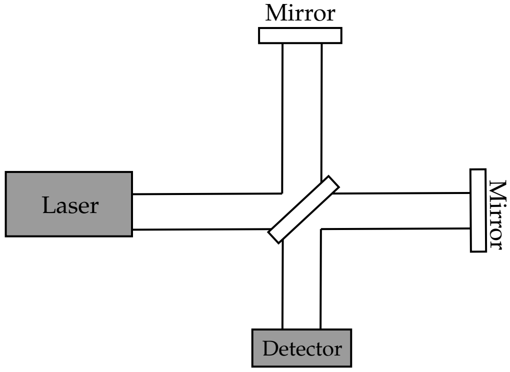
$$c^2 d\tau^2 = c^2 dt^2 - \left(\delta_{ij} + h^{\mathrm{TT}}_{ij}\right)\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}d\tau^2. \tag{1.99}$$

For a test mass initially at rest, $\frac{dx^i}{d\tau} = 0$ at all times. Then, in the TT gauge, the proper time τ measured by a clock sitting on a test mass initially at rest is the coordinate time t. Moreover, proper distances change. Consider the distance between $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$. The coordinate distance $L = x_2 - x_1$ is invariant. However, the proper distance after the passage of the wave is

$$s = \int_{x_1}^{x_2} dx \sqrt{g_{xx}} = \sqrt{g_{xx}} L = \sqrt{1 + h_+ \cos(\omega t)} L$$
$$\simeq \left(1 + \frac{1}{2} h_+ \cos(\omega t)\right) L. \quad (1.100)$$

Since the induced metric on the path g_{xx} does not depend upon x.

Therefore, the proper distance changes periodically in time because of the gravitational wave. This is the basis of interferometry, where test masses are mirrors, and proper time determines the time taken by light to make a round trip.



Interferometer

The Earth-based laboratory detector frame

An experimentalist on Earth is not using the TT frame. In a laboratory, one fixes an origin, and one uses a rigid ruler to define coordinates (under the passage of a wave, the ruler does not change its relative length, that is $\Delta L/L \ll h$). In such a frame, we expect that a mass free to move in some direction will be displaced by the passage of the gravitational wave.

Let us first think on a space-based laboratory. Conceptually, the simplest laboratory to analyze is one inside of a drag-free satellite, so that the apparatus is in free fall in the total gravitational field, including both the Earth and the gravitational wave. For such an apparatus, the metric is approximately flat:

$$ds^2 \simeq -c^2 dt^2 + \delta_{ij} dx^i dx^j. \quad (1.101)$$

We can build such a freely-falling frame using Fermi coordinates. To linear order in x^i, there is no correction to this metric because $\partial_\alpha g_{\mu\nu}\big|_P = 0$ around the origin P, where we expand. At second order, there is a correction proportional to the Riemann tensor at P:

$$ds^2 \simeq -c^2dt^2\left(1 + R_{0i0j}\big|_P\, x^i x^j\right) - 2cdtdx^i\left(\frac{2}{3}\, R_{0jik}\Big|_P\, x^j x^k\right) + dx^i dx^j\left(\delta_{ij} - \frac{1}{3}\, R_{ikjl}\Big|_P\, x^k x^l\right). \qquad (1.102)$$

Let $L_{\rm B}$ be the typical scale of variation of the metric so that the Riemann tensor is of order $R_{\mu\nu\alpha\beta} = \mathcal{O}(1/L_{\rm B}^2)$, then corrections to the metric scale as $\mathcal{O}(r^2/L_{\rm B}^2)$.

For an Earth-bound detector, the frame is not in free-fall: there is an acceleration $\vec{a} = -\vec{g}$ with respect to a local inertial frame. Furthermore, it rotates relatively to local gyroscopes (as illustrated by the Foucault pendulum) with angular velocity $\vec{\Omega}$. The metric in this laboratory frame can be found by explicitly writing the coordinate transformation from the inertial frame to the frame which is accelerating and rotating, and transforming the metric accordingly. The result, up to order $\mathcal{O}(r^2)$ is

$$\begin{aligned} ds^2 \simeq & -c^2dt^2\left(1 + \frac{2}{c^2}\vec{a}\cdot\vec{x} + \frac{1}{c^4}\left(\vec{a}\cdot\vec{x}\right)^2 - \frac{1}{c^2}\left(\vec{\Omega}\times\vec{x}\right)^2 + R_{0i0j}x^i x^j\right) \\ & + 2cdtdx^i\left(\frac{1}{c}\varepsilon_{ijk}\Omega^j x^k - \frac{2}{3}R_{0jik}x^j x^k\right) \\ & + dx^i dx^j\left(\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l\right), \end{aligned} \qquad (1.103)$$

where a^i is the acceleration of the laboratory with respect to a local free-falling frame and Ω^i is the angular velocity of the laboratory with respect to local gyroscopes. All terms involving $R_{\mu\nu\alpha\beta}$ give effects of gravitational backgrounds and gravitational waves. Moreover, the term $\frac{2}{c^2}\vec{a}\cdot\vec{x}$ corresponds to the inertial acceleration, $\frac{1}{c^4}\left(\vec{a}\cdot\vec{x}\right)^2$ is the gravitational redshift and $\frac{1}{c^2}\left(\vec{\Omega}\times\vec{x}\right)^2$ is the Lorentz time dilatation due to the angular velocity, the $\frac{1}{c}\varepsilon_{ijk}\Omega^j x^k$ is known as the *Sagnac effect*.

This metric is in the *proper detection frame on Earth*. It is implicitly used by experimentalists. At zeroth order in $r/L_{\rm B}$, the metric reduces to the flat metric. If we focus on regions smaller than the variation scale of the background, and if velocities in the apparatus are small compared to c, we can use Newtonian physics. All the effects of gravitational waves should be understandable in terms of Newtonian forces! This is to be constrasted with the TT gauge, where gravitational waves are always present in the background spacetime, and where there is no expansion in $r/L_{\rm B}$. In the detector proper frame, there are corrections linear in $r/L_{\rm B}$ that can be described in terms of Newtonian forces. Writing the geodesic equation on the metric

(1.103), and neglecting $\mathcal{O}(r^3)$ terms, we get

$$\frac{d^2x^i}{d\tau^2} = -a^i - 2\left(\vec{\Omega}\times\vec{v}\right)^i + \frac{f^i}{m} + \mathcal{O}(x^i), \tag{1.104}$$

where a^i corresponds to Newtonian gravity, $\left(\vec{\Omega}\times\vec{v}\right)^i$ is the Coriolis acceleration, f^i/m are the external forces (*e.g.* suspension mechanism). Gravitational wave effects are lower-order effects. To isolate the effects of gravitational waves, we have to focus on the response of the detector on a specific frequency window (see Fig. 1.4). The acceleration $\vec{a}$ is compensated by the suspension mechanism and all effects (called the seismic noise and Newtonian noise) are typically slowly varying (i.e. they have a frequency < 10 Hz). If we neglect such effects, we are back to the metric in the freely falling frame with only terms proportional to the Riemann tensor. It is understood that we restrict the analysis of the motion of the components $x^i(\tau)$ in the directions in which test masses (mirrors) are left free to move by the suspension mechanism, and that we only consider the Fourier components of the motion, in a frequency window where the detector is sensitive to gravitational waves. In this frequency window, we assume that time-varying Newtonian gravitational forces are sufficiently small, so that only gravitational waves contribute to the Riemann tensor. Other sources of noise include the quantum noise of the lasers at higher frequencies (above 10^5Hz). The "sweet spot" where Earth-based observations are currently possible lies in the range 10Hz to 3000Hz for LIGO-Virgo-Kagra and 1Hz to 3000Hz for the Einstein Telescope (with increased sensitivity in the $1-10$ Hertz band due to placing the detector underground).

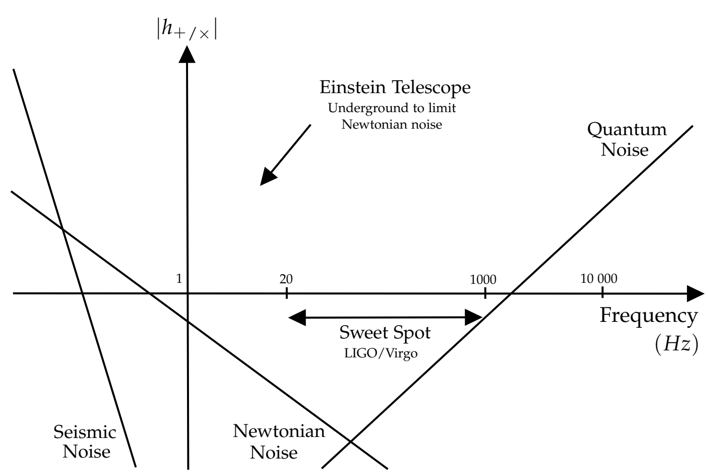


Figure 1.4: Frequency-dependent response function for a typical Earth-based detector.

Let us now derive the effective Newtonian force acting on test masses on a Earth-based detector due to gravitational waves. We start with the geodesic deviation equation written in the form

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{1.105}$$

At point P, the Christoffel symbols vanish. Since the motion of the detector is non-relativistic, *i.e.* $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau}$, terms proportional to $\frac{dx^i}{d\tau}$ can be neglected. Thus, we find

$$\frac{d^2\xi^i}{d\tau^2} + \xi^\sigma\partial_\sigma\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0. \tag{1.106}$$

Moreover, at linear order in $h_{\mu\nu}$, $\tau = t$, so $x^0 = ct = c\tau$, which implies that $\frac{dx^0}{d\tau} = c$. Since we evaluate the equation at P, we set $x^i = 0$ and $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^i x^j)$. A non-zero contribution only comes from terms in which the two derivatives on the metric present in $\partial_\sigma\Gamma^i_{00}$ are both spatial derivatives, and act on $x^i x^j$, so that

$$\xi^\sigma\partial_\sigma\Gamma^i_{00} = \xi^j\partial_j\Gamma^i_{00}. \tag{1.107}$$

We also know that

$$R^i_{0j0} = \partial_j\Gamma^i_{00} - \cancel{\partial_0\Gamma^i_{0j}} + \mathcal{O}(\Gamma\Gamma), \tag{1.108}$$

where the time-derivative part vanishes since the metric is proportional to $x^i x^j$, and $x^i = 0$ at point P. This finally yields

$$\boxed{\frac{d^2\xi^i}{dt^2} = -R^i_{0j0}\xi^j.} \tag{1.109}$$

Let us now use an incredible trick. As we saw, the Riemann tensor is invariant under linear coordinate changes, in linearized Einstein gravity. Therefore, we can use its expression in *any* frame. We know its expression in the TT frame, where the metric is the simplest. We find

$$R^i_{0j0} = R_{i0j0} = -\frac{1}{2c^2}\ddot{h}^{\mathrm{TT}}_{ij}, \tag{1.110}$$

where dots stand for coordinate time derivatives. We can then substitute this expression to find the acceleration of masses in the Earth-based detector frame. In conclusion, the equation of geodesic deviation in the Earth-based detector frame is

$$\boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}\xi^j.} \tag{1.111}$$

In the Earth-based detector frame, the effect of gravitational waves on a point particle of mass m can be described in terms of a Newtonian force

$$F_i = \frac{m}{2}\ddot{h}^{\mathrm{TT}}_{ij}\xi^j. \tag{1.112}$$

The experimenter can use Newtonian physics! The expression of the force uses the metric in the TT frame where gravitational waves are best computed. This is the best of both worlds.

Remark 1.3.2. *We used the geodesic deviation equation, which assumed that $|\xi^i|$ is much smaller than the typical scale, the reduced wavelength $\lambdabar_{\rm GW}$, over which the gravitational field changes substantially. For a detector of size L, we assumed $L \ll \lambdabar_{\rm GW}$. It is obeyed by resonant bar detectors, ground-based detectors but not space-based detectors, where a general relativistic description is necessary.*

Motion of test masses

We can now study the motion of test masses in the detector frame. Let us consider a ring of test masses initially at rest in the Earth detector frame. We fix the origin at the center of the ring. In this manner, ξ^i describes the distance with respect to the origin. This distance is the proper distance and coordinate distance since they are the same close to $x^i = 0$ in the proper detector frame. We place the ring on the (x, y)-plane at $z = 0$. We consider a gravitational wave propagating in the z direction. Considering the origin of time such that $h_{ij}^{\rm TT} = 0$, at $t = 0$. We have

$$h_{ij}^{\rm TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \sin\left(\omega(t - \cancel{\frac{z}{c}})\right), \tag{1.113}$$

z being set to 0. If a particle is initially at rest at $z = 0$, it will remain at $z = 0$ at all times from

$$\ddot{\xi}^z = \frac{1}{2}\ddot{h}_{zj}^{\rm TT}\xi^j = 0. \tag{1.114}$$

Therefore, gravitational waves are transverse not only in the sense that $\partial_i h_{ij} = 0$, but also from their physical effect: they displace test masses transversely with respect to their direction of propagation, in the case of test masses initially at rest.

Let us study the motion in the (x, y)-plane. For the + polarization, we have that

$$h_{ab}^{\rm TT} = h_+ \sin(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a, b = x, y. \tag{1.115}$$

We also consider

$$\xi_a(t) = (x_0 + \delta x(t), y_0 + \delta y(t)), \tag{1.116}$$

where x_0 and y_0 are the unperturbed positions and $\begin{cases} \delta x \ll x_0, \\ \delta y \ll y_0 \end{cases}$.

In order to find the deformation of the ring, we need to solve Eq. (1.111) for this ξ_a. We have

$$\begin{cases} \delta\ddot{x} = -\frac{h_+}{2}\omega^2\,(x_0+\delta x)\sin(\omega t), \\ \delta\ddot{y} = +\frac{h_+}{2}\omega^2\,(y_0+\delta y)\sin(\omega t). \end{cases} \tag{1.117}$$

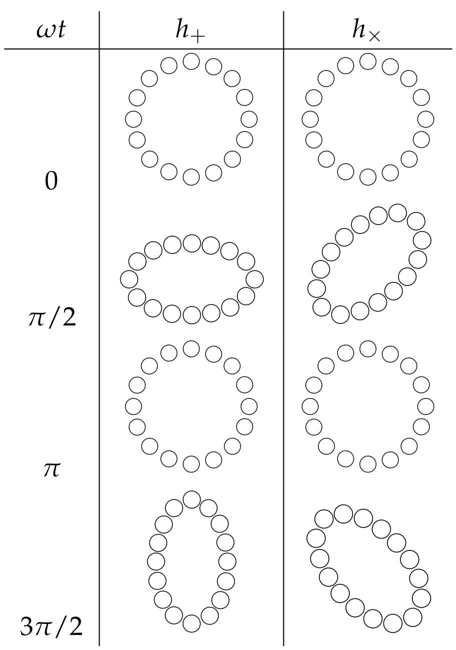
Since the perturbations δx and δy are small with respect to their unperturbed position, we can neglect them at linear order in h. Thus, we solve those equations by integrating twice with respect to time and we obtain

$$\begin{cases} \delta x(t) = +\frac{h_+}{2}x_0\sin(\omega t), \\ \delta y(t) = -\frac{h_+}{2}y_0\sin(\omega t). \end{cases} \tag{1.118}$$

For the $\times$ polarization, we proceed the same way, such that

$$\begin{cases} \delta x(t) = +\frac{h_\times}{2}y_0\sin(\omega t), \\ \delta y(t) = -\frac{h_\times}{2}x_0\sin(\omega t). \end{cases} \tag{1.119}$$

The resulting deformations can be summarized in the following table:



The Newtonian force is divergence-free:

$$\partial_i F_i = \frac{m}{2}\ddot{h}^{\mathrm{TT}}_{ij}\delta_{ij} = 0, \tag{1.120}$$

so that the area of the disk is conserved in time.

These patterns justify the names of the perturbations h_+ and $h_\times$. We see a quadrupolar pattern, rooted in the helicity 2 nature of the graviton. At each time, the pattern is invariant under a rotation of angle $2\pi/2 = \pi$. The denominator 2 in the last expression is exactly due to the helicity 2 of the graviton, see also Eq. (1.75).

1.4 Generation of Gravitational Waves

The linearized Einstein equations in the de Donder gauge read

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \quad ; \quad \partial^\mu\bar{h}_{\mu\nu} = 0 \quad ; \quad \partial^\nu T_{\mu\nu} = 0. \tag{1.121}$$

We can solve this equation in terms of the retarded Green function

$$\Box G(x - x') = \delta^{(4)}(x - x'), \tag{1.122}$$

where

$$\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\int d^4x\, G(x - x')T_{\mu\nu}(x'). \tag{1.123}$$

The Green function is given explicityly by

$$G(x - x') = -\frac{1}{4\pi|\vec{x} - \vec{x}\,'|}\delta\left(x^0_{\text{ret}} - x'^0_{\text{ret}}\right), \tag{1.124}$$

where $x'^0 = ct'$, $x^0_{\text{ret}} = ct_{\text{ret}}$ and $t_{\text{ret}} = t - |\vec{x} - \vec{x}\,'|/c$. Outside the source, we can use TT gauge and we have

$$h^{\text{TT}}_{ij} = \Lambda_{ij,kl}h_{kl}, \tag{1.125}$$

where h_{kl} is in harmonic gauge. Therefore, outside the source, we have

$$\boxed{h^{\text{TT}}_{ij}(t,\vec{x}) = \frac{4G}{c^4}\Lambda_{ij,kl}(\vec{x})\int d^3x\,\frac{1}{|\vec{x} - \vec{x}\,'|}T_{kl}\left(t - \frac{|\vec{x} - \vec{x}\,'|}{c}, \vec{x}\,'\right).} \tag{1.126}$$

Note that h^{TT}_{ij} depends upon integrals of spatial components of T_{kl}. The temporal components of $T_{\mu\nu}$ are related by the conservation law $\partial^\nu T_{\mu\nu} = 0$.

Let us do some geometry to simplify the expression (1.126). We set the origin at the center of the source, which has dimension d. Thus, $|\vec{x}\,'| \leq d$. The setup considered is depicted on Fig. 1.5:

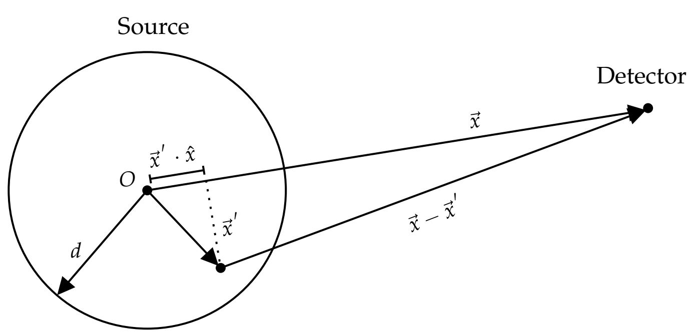


Figure 1.5: Kinematics of the source with respect to the detector.

Let $\hat{x} = \vec{x}/r$ be the unit vector from the source towards the detector (which encodes the angle of sight of the source with respect to the detector), and $r = |\vec{x}|$ the distance between the source and the detector. Astrophysical compact binary sources in the last stages of the inspiral obey $d \ll r$ since distances are $r \gg 1$ Mpc while the source size is of the order of magnitude of a couple to a hundred of Schwarzschild radii of the largest compact body (= 3km for a solar mass compact body). We have

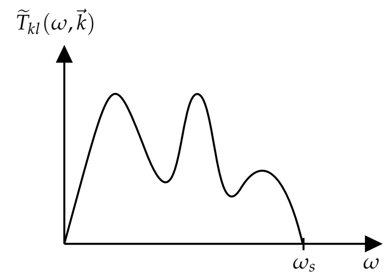
$$|\vec{x} - \vec{x}\,'| = r - \vec{x}\,' \cdot \hat{x} + \mathcal{O}(d^2/r), \tag{1.127}$$

where higher orders terms are of order d^2/r and can be safely neglected. Therefore, at large distance from the source,

$$h_{ij}^{\mathrm{TT}}(t,\vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{x}) \int d^3x'\, T_{kl}\left(t - \frac{r}{c} + \frac{\vec{x}\,'\cdot\hat{x}}{c}, \vec{x}\,'\right). \tag{1.128}$$

We see that h_{ij}^{TT} is proportional to the inverse distance to the source. This is analogous to electromagnetism: radiation fields admit a gauge potential that is also inversely proportional to the distance to the source of electromagnetic radiation (in contrast to a static Coulombic potential without radiation which falls off as r^{-2}). It is important to keep the dependency in x' at leading order in the integral in order to capture the leading order effects of the source. To compute this integral, we first perform a Fourier transform,

$$T_{kl}\left(t - \frac{r}{c} + \frac{\vec{x}\,'\cdot\hat{x}}{c}, \vec{x}\,'\right) = \int \frac{d^4k}{(2\pi)^4}\widetilde{T}_{kl}(\omega,\vec{k}) e^{-i\omega\left(t-\frac{r}{c}+\frac{\vec{x}\,'\cdot\hat{x}}{c},\vec{x}\,'\right)+i\vec{k}\cdot\vec{x}\,'}. \tag{1.129}$$



We shall now assume the post-Newtonian approximation, $v \ll c$, where $v = \omega_s d$ is the typical velocity of the source and ω_s the typical source frequency. More precisely, we assume that the source has Fourier modes with frequencies bounded from above, $\omega < \omega_s$.

Then, we have that

$$\omega\frac{\vec{x}'\cdot\hat{x}}{c} \leq \frac{\omega_s d}{c} \ll 1, \tag{1.130}$$

which allows us to expand the exponential in orders of ω/c:

$$e^{-i\omega\left(t-\frac{r}{c}+\frac{\vec{x}\,'\cdot\hat{x}}{c}\right)} = e^{-i\omega\left(t-\frac{r}{c}\right)}\left[1 - i\frac{\omega}{c}\vec{x}\,'\cdot\hat{x} + \mathcal{O}\left(\frac{\omega}{c}\right)^2\right]. \tag{1.131}$$

This is equivalent, in position space, to expanding as follows:

$$T_{kl}\left(t - \frac{r}{c} + \frac{\vec{x}\,'\cdot\hat{x}}{c}, \vec{x}\,'\right) \simeq T_{kl}\left(t - \frac{r}{c}, \vec{x}'\right) + \frac{\vec{x}'\cdot\hat{x}}{c}\partial_t T_{kl} + \mathcal{O}\left(\partial_t^2 T_{kl}\right). \tag{1.132}$$

We define the multipoles of the stress tensor as

$$S_{ij}(t) = \int d^3x\, T_{ij}(t,\vec{x}), \tag{1.133}$$

$$S_{ij,k}(t) = \int d^3x\, T_{ij}(t,\vec{x})x_k, \tag{1.134}$$

$$S_{ij,kl}(t) = \int d^3x\, T_{ij}(t,\vec{x})x_k x_l. \tag{1.135}$$

We can now therefore resolve the spatial integrals in terms of the multipoles of the stress-tensor and express the metric perturbation in TT gauge as

$$h_{ij}^{\mathrm{TT}}(t,\vec{x}) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})\left[S_{kl}(t) + \frac{1}{c}\hat{x}_m \dot{S}_{kl,m}(t) + \frac{1}{2c^2}\hat{x}_m\hat{x}_p\ddot{S}_{kl,mp}(t) + \dots\right]_{\mathrm{ret}}, \tag{1.136}$$

evaluated at the retarded time $t - r/c$. The linear term in $\hat{x}$ is proportional to $\omega_s d/c \sim v/c$, and so on. Therefore, weak sources with non-relativistic velocities emit gravitational radiation that is determined by the lowest spatial multipole moment of the source. Higher order multipole moments bring corrections in powers of v/c: they are post-Newtonian (PN) corrections. Using the conservation of the stress-energy tensor, it is possible to convert S_{ij} in terms of moments of T_{00}:

$$M_{ij} = \frac{1}{c^2}\int d^3x\, T_{00}x_i x_j, \tag{1.137}$$

which is the *quadrupole moment*. One can prove that $S_{ij} = \ddot{M}_{ij}/2$ (see exercise session). Finally, one obtains

$$\boxed{h_{ij}^{\mathrm{TT}}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c}\Lambda_{ij,kl}(\hat{x})\ddot{M}_{kl}\left(t - \frac{r}{c}\right) + \text{PN corrections.}} \tag{1.138}$$

We conclude that gravitational waves are generated by the acceleration of mass quadrupoles. We also learned that we do not need to know much about the source to compute its gravitational wave emission as long as the source is non-relativistic and the gravitational field around the sources is weak: we only need to know its lower-order multipole moments. In the case of mergers of compact bodies, the motion is relativistic and takes place within the strong field region of gravitational interaction. In that case, the approximations that we made are not valid, and other analytic (e.g. self-force) or numerical schemes (e.g. numerical relativity) are necessary to compute the gravitational wave emission.

The Energy of Gravitational Waves

Defining the energy of gravitational waves is a very subtle point. It is clear that gravitational waves carry energy-momentum because they accelerate masses. In 1961, Bondi wrote an influencial *Nature* paper [5] proving that gravitational waves carry energy, which led to the definition of the *Bondi mass* at $\mathscr{I}^+$, the null boundary of asymptotically flat spacetimes where outgoing radiation flows. That milestone ended a multi-decade long debate on the physical nature of gravitational waves. This approach is based on the non-linear theory of General Relativity with boundary conditions in the asymptotic region and it

[5] H. Bondi. Gravitational Waves in General Relativity. *Nature*, 186(4724): 535–535, 1960

is now called the Bondi-Metzner-van den Burg-Sachs or BMS framework. In full General Relativity (non-perturbatively), the energy is only defined as a codimension two integral at fixed time and radius, far from the sources. This is rooted in the holographic nature of gravity: the Hamiltonian is a surface term. Such considerations go well beyond this introductory class. In what follows we will instead present the definition of energy within the perturbative expansion of the metric.

According to General Relativity, any form of energy induces curvature. In linearized gravity, the total energy is conserved as a consequence of Noether's theorem, and it is not related to the gravitational waves contained in the linearized metric. Indeed, the linear field obtained by perturbing the Schwarzschild metric

$$g^{\delta M}_{\mu\nu} dx^\mu dx^\nu = -(1 - \frac{2GM}{c^2 r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.139}$$

as $M \mapsto M + \delta M$,

$$h^{\delta M}_{\mu\nu} dx^\mu dx^\nu = \frac{2G\delta M}{c^2 r} dt^2 + \frac{2G\delta M}{c^2 r} dr^2 \tag{1.140}$$

adds a contribution to the mass δM where δM can be freely specified independently of any other modes present in the linear metric. Linear fields are just a superposition of modes without any influence between them.

At second order in perturbation theory however, modes start to couple to one another. Gravitational waves backreact and produce curvature, which allows to define energy as an integral of a suitable function of that curvature over an hypersurface at fixed time. In perturbative General Relativity, thanks to Minkowski acting as a background structure, one can still use standard tools of field theory along with the Noether theorem to define symmetries and associated conserved quantities.

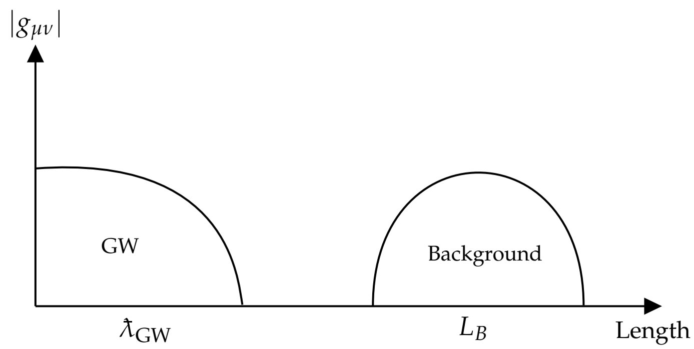
However, there is yet another catch. It turns out that the perturbation scheme

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots \tag{1.141}$$

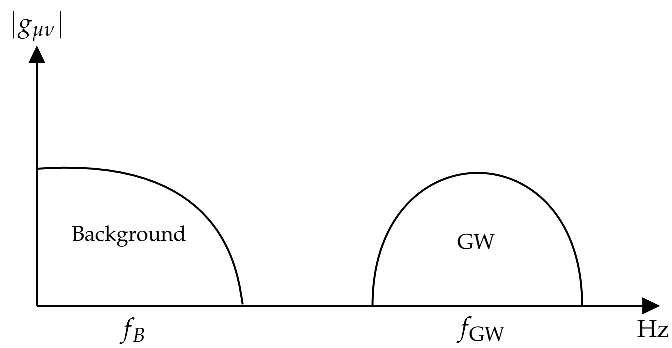
is inconsistent, as we will demonstrate, because it excludes that gravitational waves curve the spacetime. Instead, we must allow for a dynamical background (at least containing a varying total energy):

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) + h^{(2)}_{\mu\nu}(x) + \dots, \quad |h_{\mu\nu}| \ll 1. \tag{1.142}$$

The definition of what is the background and what is the perturbation should be made unambiguously. In what follows, we will distinguish the notion of background and perturbation through their frequency content: the background will be the low frequency content, while the perturbation will be the high frequency content. This is similar to splitting sea waves into a incoherent superposition of waves forming the background, and the localized perturbations of interest. More precisely, we consider the situation in which, in some reference frame, we can separate the metric into a background plus fluctuations, where the separation is based on a scale in either time or space. If the splitting is valid in position space, we have $\lambda\!\!\!^{-}_{\mathrm{GW}} \ll L_B$:



If the splitting is valid in frequency space, we have $f_B \ll f_{\mathrm{GW}}$:



As we discussed earlier, see Figure 1.4, ground-based detectors indeed obey $f_B \ll f_{\mathrm{GW}}$.

As a consequence of this split, we have two small parameters at our disposal:

$$(1) \quad h = |h_{\mu\nu}|, \qquad (2) \quad \frac{\lambda\!\!\!^{-}_{\mathrm{GW}}}{L_B} \quad \text{or} \quad \frac{f_B}{f_{\mathrm{GW}}}. \tag{1.143}$$

In order to define the energy we need to find how gravitational waves curve the background. Let us expand Einstein's equations to quadratic order in $h_{\mu\nu}$, starting from the convenient form

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{1.144}$$

We expand

$$R_{\mu\nu}[g] = \overline{R}_{\mu\nu}[\overline{g}] + R^{(1)}_{\mu\nu}[h;\overline{g}] + R^{(2)}_{\mu\nu}[h,h;\overline{g}] + \dots \tag{1.145}$$

The first term contains low frequency modes, the second high frequency modes, and the last one contains both modes†. Einstein's equations can be split into

$$\text{low mode eq.: } \overline{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{low}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{low}}, \tag{1.146}$$

$$\text{high mode eq.: } R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{high}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{high}}. \tag{1.147}$$

† This is simply because $e^{\vec{k}\cdot\vec{x}}e^{\vec{k}\cdot\vec{x}} \sim e^{2\vec{k}\cdot\vec{x}}$ and $e^{\vec{k}\cdot\vec{x}}e^{-\vec{k}\cdot\vec{x}} \sim e^0$. In words, high modes times high modes give either high modes or low modes.

The high-mode equation is essentially the linear equation that we already considered for gravitational wave generation, corrected with second order perturbations. The main interest lies in the low-mode equation.

Far from sources where $T^{\mu\nu} \sim 0$, we have $\overline{R}_{\mu\nu} \sim [R^{(2)}_{\mu\nu}]^{\text{low}}$. Since we have two small parameters, this is possible to equate two different perturbative orders in h. On the one hand, we have $\overline{R}_{\mu\nu} \sim \partial^2\overline{g}_{\mu\nu} \sim \frac{1}{L_B^2}$, where L_B is the background wavelength. On the other hand, we have

$$[R^{(2)}_{\mu\nu}]^{\text{low}} \sim \partial h \partial h + \partial^2 h^2 \sim \frac{h^2}{\lambda\!\!\!^{-2}}, \tag{1.148}$$

since partial derivatives acts on waves. We deduce that

$$h^2 \sim \left(\frac{\lambda\!\!\!^-}{L_B}\right)^2 \quad \Leftrightarrow \quad \boxed{h \sim \frac{\lambda\!\!\!^-}{L_B},} \tag{1.149}$$

far from sources. Thererfore, if we push up the scale L_B as $L_B \to \infty$, the perturbation amplitude h will go to zero, $h \to 0$, which runs the perturbative description of gravitational waves: the perturbative scheme does not apply at all.

If $T_{\mu\nu}$ dominates $[R^{(2)}_{\mu\nu}]^{\text{low}}$, we find $\frac{1}{L_B^2} \sim \frac{h^2}{\lambda\!\!\!^{-2}} + \text{matter} \gg \frac{h^2}{\lambda\!\!\!^{-2}}$. Therefore, one has

$$h \ll \frac{\lambda\!\!\!^-}{L_B}. \tag{1.150}$$

In both cases, we cannot set $L_B \to \infty$. That would fix $\overline{g}_{\mu\nu} = \eta_{\mu\nu}$ but also $h = 0$. This explains why the linearized expansion is fundamentally inconsistent if expanded around a Minkowski metric.

The low-mode equation allows to define the energy-momentum of gravitational waves. It can be obtained practically as follows. We introduce an intermediate time scale $\bar{t}$:

$$\frac{1}{f_B} \gg \bar{t} \gg \frac{1}{f_{\text{GW}}}. \tag{1.151}$$

Then, we average over $\bar{t}$, *i.e.* over many periods of gravitational waves:

$$\boxed{\overline{R}_{\mu\nu} = -\left\langle R^{(2)}_{\mu\nu} \right\rangle + \frac{8\pi G}{c^4}\left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right\rangle .} \tag{1.152}$$

This was understood in the sixties. This is a renormalization group flow: we integrated out the high frequencies to describe the physics of low frequencies. We define the low-frequency part of the stress-tensor as $\overline{T}^{\mu\nu} = \langle T^{\mu\nu}\rangle$. In perturbation theory, we have approximately, $\left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right\rangle \simeq \bar{T}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\bar{T}$.

Typically, the stress-energy tensor $T^{\mu\nu}$ generated by a macroscopic matter distribution is smooth: . Otherwise, we can define the low-frequency . It is convenient to add traces and define

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}R^{(2)} \right\rangle , \tag{1.153}$$

where $R^{(2)} = \overline{g}^{\mu\nu}R^{(2)}_{\mu\nu}$. Its trace is thus

$$t = \overline{g}^{\mu\nu}t_{\mu\nu} = \frac{c^4}{8\pi G}\left\langle R^{(2)} \right\rangle . \tag{1.154}$$

We find from these two definitions that

$$-\left\langle R^{(2)}_{\mu\nu} \right\rangle = \frac{8\pi G}{c^4}\left(t_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}t \right) . \tag{1.155}$$

Plugging it into (1.152), we find

$$\boxed{\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \frac{8\pi G}{c^4}\left(\overline{T}_{\mu\nu} + t_{\mu\nu}\right) ,} \tag{1.156}$$

Bianchi's identities imply $\overline{D}^{\mu}\overline{T}_{\mu\nu} = 0$ which implies in turn that $\overline{D}^{\mu}t_{\mu\nu} = 0$.

where $\overline{T}_{\mu\nu}$ is the low frequency matter and $t_{\mu\nu}$ is quadratic in h. This is the coarsed-grained form of Einstein's equations. These equations determine the low frequency dynamics of $\overline{g}_{\mu\nu}$. There is no fundamental distinction between a background metric and fluctuations over it in the sense that the gravitational field is the entire metric. We can define however an effective low frequency stress-energy tensor of gravitational waves using a coarsed-grained macroscopic description.

Let us compute $t_{\mu\nu}$ in the spacetime region far from the source, where the background spacetime can be considered flat up to $\mathcal{O}(1/r)$ corrections. This region encloses the detector. Mathematically, this region is located close to null infinity $\mathscr{I}^+$. Using the explicit expressions and replacing $\overline{D}_\mu$ par ∂_μ, we get

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\Big[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu}$$
$$-h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha_\nu\partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha_\nu\partial_\alpha h_{\beta\mu}$$
$$-\partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu} + \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu}$$
$$+\frac{1}{2}\partial^\alpha h\partial_\nu h_{\alpha\mu} + \frac{1}{2}\partial^\alpha h\partial_\mu h_{\alpha\nu}\Big]. \tag{1.157}$$

Out of the ten components of $h_{\mu\nu}$, eight are gauge modes and two are physical modes. We gauge fix to harmonic gauge $\overline{D}^\mu\overline{h}_{\mu\nu} = \partial^\mu\overline{h}_{\mu\nu} = 0$ up to negligible corrections. We can, in addition, gauge fix $h = 0$, which implies $\overline{h}_{\mu\nu} = h_{\mu\nu}$. We are left with the physical modes h^{TT}_{ij} and pure gauge modes $h_{\mu\nu} = \mathcal{L}_\xi\eta_{\mu\nu}$ where the infinitesimal generator of diffeomorphims ξ^μ obeys $\Box\xi^\mu = 0$.

We can now drastically simplify $R^{(2)}_{\mu\nu}$. Let us consider for definiteness the case of a spatial (wavelength) average $\langle\ \rangle$. In that case, any spatial derivative ∂_i can be freely integrated by parts: all boundary terms vanish since $h_{\mu\nu} \sim 1/r$. What about the temporal derivatives ∂_0 acting on the linearized metric $h_{\mu\nu}$? Since it obeys $\Box h_{\mu\nu} = 0$, the linear metric is a linear combination of plane waves. Time derivatives can be substituted in terms of spatial derivatives by combining the two properties

$$\begin{cases}\partial_0 e^{ik\cdot x} = ik_0 e^{ik\cdot x}\\ \partial_i e^{ik\cdot x} = ik_i e^{ik\cdot x},\end{cases} \tag{1.158}$$

with $k_0 = |k_i|$. Indeed, we can find a rotation such that the wave is aligned along the z direction: $\Lambda^i_j k^j = (0,0,k^z)$. Then

$$\Lambda^i_j\partial_j e^{ik\cdot x} = (0,0,1)\partial_0 e^{ik\cdot x} \quad\Leftrightarrow\quad \partial_0 e^{ik\cdot x} = \Lambda^z_j\partial_j e^{ik\cdot x}. \tag{1.159}$$

As a summary, any derivative ∂_μ can be freely integrated by parts inside the average $\langle\cdot\rangle$ in the case of a spatial average. The argument applies similarly in the case of a temporal (frequency) average. In that case, time derivatives ∂_0 can be integrated by parts directly while spatial derivatives are expressed in terms of time derivatives as $\partial_i(\cdot) = k_i/k_0\partial_0(\cdot)$ and then ∂_0 is integrated by parts.

Using $\partial^\mu h_{\mu\nu} = 0$ and $\Box h_{\mu\nu} = 0$, we obtain

$$\left\langle R^{(2)}_{\mu\nu}\right\rangle = -\frac{1}{4}\left\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle, \tag{1.160}$$
$$\left\langle R^{(2)}\right\rangle = 0. \tag{1.161}$$

All in all, this yields

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle.} \tag{1.162}$$

Let us check that the residual gauge modes $\mathcal{L}_\xi \eta_{\mu\nu}$ do not contribute to $t_{\mu\nu}$. Under a gauge transformation, we have

$$\begin{aligned} \delta_\xi t_{\mu\nu} &= \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu(\delta_\xi h^{\alpha\beta}) + (\mu \leftrightarrow \nu)\right\rangle \\ &= \frac{c^2}{16\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu \partial^\alpha \xi^\beta + (\mu \leftrightarrow \nu)\right\rangle \\ &= 0, \end{aligned} \tag{1.163}$$

$\delta h^{\alpha\beta} = \partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha$.

after integrating by parts and using $\partial^\mu h_{\mu\nu} = 0$. We can then use TT modes and we have

$$t^{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial^\mu h_{ij}^{\mathrm{TT}} \partial^\nu h_{ij}^{\mathrm{TT}}\right\rangle. \tag{1.164}$$

The gauge invariant energy density is

$$t^{00} = \frac{c^2}{32\pi G}\left\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}}\right\rangle, \tag{1.165}$$

where the dot correponds to a derivative with respect to t: $\partial_t = c\partial_0$. In terms of amplitudes h_+ and $h_\times$, $h_{ij} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}$, which gives

$$t^{00} = \frac{c^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle. \tag{1.166}$$

The Bianchi identity implies $\partial^\mu t_{\mu\nu} = 0$. Having obtained the energy-impulsion tensor carried by gravitational waves, it is now straightforward to obtain the energy flux: the energy of a gravitational wave flowing per unit time through a unit surface at a large distance from the source. We have

$$0 = \int_V d^3x\, \partial_\mu t^{\mu\nu} = \int_V d^3x\, \left(\partial_0 t^{0\nu} + \partial_i t^{i\nu}\right). \tag{1.167}$$

We now select the $\nu = 0$ component and take V as a large volume around the source bounded by a surface S. The gravitational wave energy inside V is

$$E_V = \int_V d^3x\, t^{00}. \tag{1.168}$$

Using $\partial_0 = c^{-1}\partial_t$, we find that

$$\frac{1}{c}\frac{dE_V}{dt} = -\int_V d^3x\, \partial_i t^{0i} = -\int_S dS\, \hat{n}_i t^{0i}, \tag{1.169}$$

with $dS = r^2 d^2\Omega$ the measure on the surface S and $\hat{n} = \hat{r}$ the outer unit normal to the surface S. Thus, we have

$$\frac{dE_V}{dt} = -c\int d^2\Omega\, r^2 t^{0r}, \tag{1.170}$$

where $t^{0r} = \frac{c^4}{32\pi G}\left\langle \partial^0 h_{ij}^{\rm TT} \frac{\partial}{\partial r} h_{ij}^{\rm TT} \right\rangle$. A gravitational wave that propagates radially outwards, at sufficiently large distance r, has the form

$$h_{ij}^{\rm TT}(t,r) = \frac{1}{r} f_{ij}\left(t - \frac{r}{c}\right) + \mathcal{O}(r^{-2}). \tag{1.171}$$

Indeed,

$$\begin{aligned}
\Box h_{ij}^{\rm TT} &= \left(-\frac{1}{c^2}\partial_t^2 + \partial_r^2\right) h_{ij}^{\rm TT} = 0 \\
\Leftrightarrow \quad &\left(-\frac{1}{c^2}\partial_t^2 + \partial_r^2\right)\left(\frac{1}{r} f_{ij}\left(t - \frac{r}{c}\right) + \mathcal{O}(r^{-2})\right) \qquad (1.172)\\
&= \mathcal{O}(r^{-2}) + \frac{1}{r}\left(-\cancel{\frac{1}{c^2} f_{ij}''} + \cancel{\frac{1}{c^2} f_{ij}''}\right), \qquad (1.173)
\end{aligned}$$

which is satisfied. Therefore,

$$\begin{aligned}
\frac{\partial}{\partial r} h_{ij}^{\rm TT}(t,r) &= \mathcal{O}(r^{-2}) + \frac{1}{r}\frac{\partial}{\partial r} f_{ij}\left(t - \frac{r}{c}\right) \\
&= \mathcal{O}(r^{-2}) + \frac{1}{r}\left(-\frac{1}{c}\right)\frac{\partial}{\partial t} f_{ij}\left(t - \frac{r}{c}\right) \\
&= -\partial_0 h_{ij}^{\rm TT} + \mathcal{O}(r^{-2}) = \partial^0 h_{ij}^{\rm TT} + \mathcal{O}(r^{-2}). \qquad (1.174)
\end{aligned}$$

At large distances, we obtain

$$t^{0r} = \frac{c^4}{32\pi G}\left\langle \partial^0 h_{ij}^{\rm TT} \frac{\partial}{\partial r} h_{ij}^{\rm TT} \right\rangle = t^{00}, \tag{1.175}$$

and

$$\frac{dE_V}{dt} = -c\int d^2\Omega\, r^2 t^{00}. \tag{1.176}$$

The energy E_V is the opposite of the gravitational wave energy which is escaping the system: $E_V = -E_{\rm GW}$. The outgoing flux of gravitational wave energy per unit angle is finally

$$\boxed{\frac{d^2 E_{\rm GW}}{d\Omega dt} = r^2 c t^{00} = \frac{c^3 r^2}{32\pi G}\left\langle \dot{h}_{ij}^{\rm TT} \dot{h}_{ij}^{\rm TT} \right\rangle.} \tag{1.177}$$

In terms of h_+ and $h_\times$, this becomes

$$\frac{d^2 E_{\rm GW}}{d\Omega dt} = \frac{c^3}{16\pi G}\left\langle (r\dot{h}_+)^2 + (r\dot{h}_\times)^2 \right\rangle \tag{1.178}$$

Analogously, we compute the flux of momentum

$$P_V^k = -\frac{1}{c}\int_V d^3x\, t^{0k}. \tag{1.179}$$

For outward-directed gravitational waves, we have

$$\begin{aligned}
c\partial_0 P_V^k &= \int_V d^3x\, \partial_0 t^{0k} \\
&= \int_S d^2\Omega\, r^2 t^{0k}, \qquad (1.180)
\end{aligned}$$

which yields

$$\frac{dP^k}{dt} = -\frac{c^3}{32\pi G} \int d^2\Omega \left\langle r\dot{h}_{ij}^{\mathrm{TT}} r \partial_k h_{ij}^{\mathrm{TT}} \right\rangle . \tag{1.181}$$

We can obtain similarly the fluxes of the Lorentz charges (angular momentum and mass moment).

Remark 1.4.1. *In conclusion, we confirmed that gravitational waves carry away energy when second order effects in perturbation theory are taken into account consistently. Energy is conserved at linear order in perturbation theory but second order effects are non-dissipative and lead to energy extraction of the system through gravitational waves. The reason why energy is conserved at linear order can be understood intuitively in several ways. Linear solutions are superpositions of wavepackets that can be chosen to be no spatial extension to infinity. The energy as computed from the* $1/r$ *term in the metric can therefore be tuned to be vanishing for any localized wavepacket. In other words, one can localize linear fields on a compact spatial support and the mass as computed at spatial infinity is zero. Once second order effects are taken into account, energy is released to null infinity. It becomes then also clear that the background cannot be Minkowski, otherwise one would take away energy starting from zero, which would be pathological. Instead, the background should contain a finite amount of energy, it should be different from Minkowski, in order to consistently define gravitational waves. This is also a result that we formally derived in this section.*

2

Theory of Gravitational Waves: waveforms

One of the main challenges of gravitational relativists is to predict the shape of the gravitational waves to be observed by current and future gravitational wave observatories to the precision set by the observational devices, in order to avoid any wrong inference of the physics due to systematic modelling errors.

We have deduced so far the main features of gravitational waves. We will now turn to the modern methods used to derive the precise gravitational waveforms emitted from the main event detectable by current observatories: compact binary mergers.

The problem is difficult to tackle due to the non-linearity of Einstein's equations. Several approximation methods have to be used, compared and complemented in order to obtain a suitable database of accurate enough gravitational waveforms. There are currently four main first-principle methods to obtain gravitational waveforms:

- Post-Newtonian/ post-Minkowskian (PN/PM) theory or weak field expansion,
- self-force theory (SF),
- numerical relativity (NR),
- effective one-body (EOB) and phenomelogical waveforms.

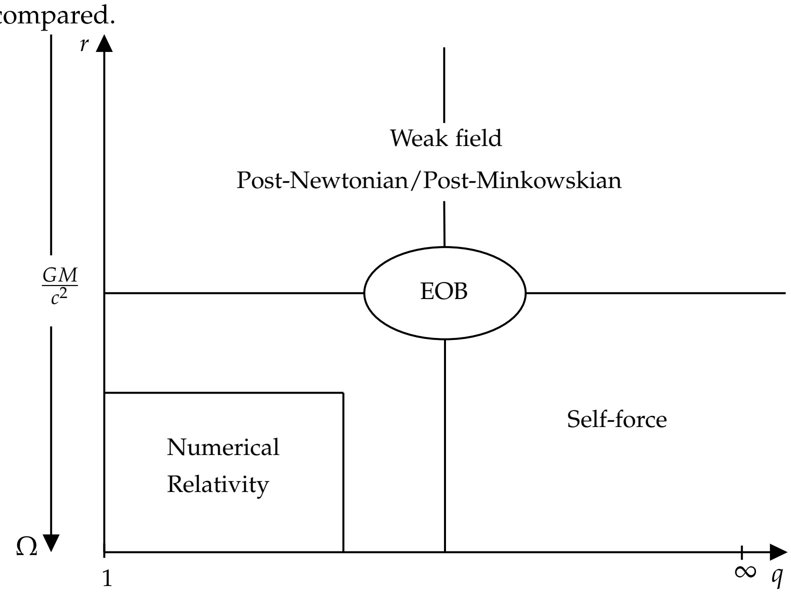
All such methods are applicable in a different range of parameters of the compact binary system. The main parameters of the system are the two masses m_1, m_2 of the bodies with $m_1 \geq m_2$ by convention and the geodesic distance r between the bodies. We can derive several auxiliary parameters. The angular velocity of the secondary body around the primary body scales as $\Omega \sim r^{-3/2}$, which is a consequence of Kepler's law. There are several standard definitions of masses:

Exercise. Solve the geodesic equation around the Schwarzschild black hole for a circular orbit. Derive that the angular velocity scales as $\Omega \sim r^{-3/2}$.

- Total mass: $M = m_1 + m_2$,

- reduced mass: $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$,
- small mass ratio: $\varepsilon = \dfrac{m_2}{m_1}$, where $0 < \varepsilon \leq 1$,
- large mass ratio: $q = \dfrac{m_1}{m_2}$, where $1 \leq q < \infty$,
- symmetric mass ratio: $\nu = \dfrac{m_1 m_2}{(m_1 + m_2)^2}$, where $0 < \nu \leq 1/4$. The equal mass case is $\nu = \frac{1}{4}$.

The range of applicability of the various methods is depicted on Figure 2. For high large mass ratio and large distances, both the weak field and the self-force approaches are applicable and can be compared.



How precise should the waveforms be derived? The answer depends upon the precision set by the detectors. Ideally, we would like to ask that the modelled waveform is indistinguishable from the target exact waveform when performing measurements with a chosen detector. From two waveforms and a detector sensitivity curve one can define the so-called unfaithfulness $\overline{\mathcal{F}}$ of two waveforms relative to the chosen detector (see part of the class on detectors). A well accepted criterion is

$$\overline{\mathcal{F}} < \frac{\mathcal{O}(1)}{SNR^2}, \tag{2.1}$$

with SNR the signal-to-noise ratio (see part of the class on data analysis). For the Einstein Telescope, the loudest events are expected to reach a very large $SNR \sim 10^3$, which requires waveform models with an unfaithfulness $\overline{\mathcal{F}} \sim 10^{-5} - 10^{-6}$. This is way beyond the capabilities of *any* current day state-of-the-art waveform model or numerical relativity simulation which have a current unfaithfulness of $\overline{\mathcal{F}} \sim 10^{-3}$. There is also a diversity of intrinsic parameters of the binary system (such as the spin of the two bodies, the eccentricity, ...) which is currently not modelled very accurately. In addition, for black hole (BH) - neutron star (NS) mergers and NS-NS mergers, a lot of modelling is required regarding dense matter at the core of neutron stars, magnetic fields, neutrino radiation, non-ideal fluid dynamics and so on. The current status of each method is as follows:

- NR: simulations exist since ~ 20 years. Seminal simulations that made history are the ones of Shibata-Uryu [1] in 2000 of NS-NS mergers and the ones of Pretorius [2] of BH-BH mergers in 2004. Hundreds of high accuracy simulations are publicly available. Several dozen of codes exist.
- PN/PM: The method has been developed since the birth of general relativity but it became systematic since the '80ies with the work of Blanchet, Damour and Will. In 2022, the 4PN gravitational phase has been computed for binaries with spinning, but non-precessing, bodies.
- Second order self-force inspiral waveforms for binaries without spin, eccentricity or inclination have been formulated in 2019.
- EOB: This method was invented in 1998 by Damour and Buonanno. Two competing models exist (TEOB & SEOB), incorporating many independent data from NR, PN/PM and SF theories.

[1] Masaru Shibata and Kōji Uryū. Simulation of merging binary neutron stars in full general relativity. *Physical Review D*, 61(6), February 2000

[2] Frans Pretorius. Numerical relativity using a generalized harmonic decomposition. *Classical and Quantum Gravity*, 22 (2):425–451, January 2005

As the reader can notice from the dates mentioned above. The development of waveforms is a topic under active recent development. In the following, we will review some of these methods based mainly on the lecture notes of Blanchet [3], Maggiore's book on Gravitational Waves [4] and Deruelle and Uzan's "Relativity in Modern Physics" book [5].

[3] Luc Blanchet. Post-newtonian theory for gravitational waves, 2024

[4] Michele Maggiore. *Gravitational Waves. Vol. 1: Theory and Experiments*. Oxford University Press, 2007

[5] Nathalie Deruelle and Jean-Philippe Uzan. *Relativity in Modern Physics*. Oxford Graduate Texts. Oxford University Press, 8 2018

2.1 *The MPN/PM formalism*

The acronym MPN/PM stands for Multipolar Post-Newtonian/Post Minkowskian formalism. We will restrict our considerations to compact binary coalescences. There are two relevant length scales: the distance d between the two bodies and the typical gravitational wavelength $\lambda\!\!\!^{-}_{\mathrm{GW}}$. For compact binaries in the non-relativistic regime, we have $d \ll \lambda\!\!\!^{-}_{\mathrm{GW}}$. Indeed, let ω_s be the typical frequency of the motion inside the source. The typical velocity of the source is $v \sim \omega_s d$. The frequency of the radiation will also be of the order of $\omega_{\mathrm{GW}} \sim 2\omega_s$, as we saw earlier. The reduced gravitational wavelength is therefore

$$\lambda\!\!\!^{-}_{\mathrm{GW}} = \frac{c}{\omega_{\mathrm{GW}}} \sim \frac{c}{\omega_s} \sim \frac{c}{v} d. \tag{2.2}$$

For a non-relativistic system $v \ll c$, which proves $d \sim \lambda\!\!\!^{-}_{\mathrm{GW}}$. This separation of scales allows to use the method of matched asymptotic expansions. One distinguishes two zones: the source zone $r \ll \lambda\!\!\!^{-}_{\mathrm{GW}}$ and the exterior zone $r \gg d$. These zones have an overlap in the matching region $d \ll r \ll \lambda\!\!\!^{-}_{\mathrm{GW}}$ that exists precisely because of the separation of scales, see Figure 2.1. In the source zone, relativistic effects are small, and a post-Newtonian (PN) expansion is valid. In the exterior zone, there is no source, and a post-Minkowskian (PM) approximation is valid. In the overlap region, both PN and PM expansions are valid. A multipolar decomposition in terms of spherical harmonics is performed to simplify the implementation of the matching conditions. It is useful because the lowest harmonics are dominant. The terminology for denoting the level of approximation of the PN approximation is as follows. We denote terms

$$n\mathrm{PN} = \mathcal{O}\left(\frac{v^{2n}}{c^{2n}}\right), \tag{2.3}$$

with 0PN being the Newtonian limit. It turns out that the near-zone (source) dynamics is conservative up to 2PN order. At 2.5PN order, there is gravitational wave emission, which is implemented in the near-zone as a radiation-reaction effect on the source.

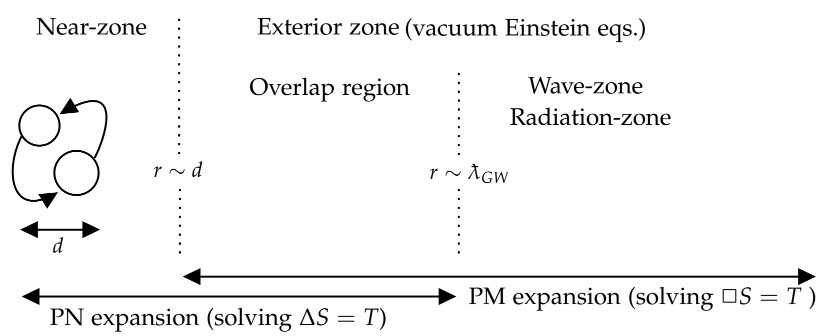


Figure 2.1: The two zones of the PN/PM formalism, with the overlap region indicated. The radiation zone lies behind radii corresponding to the typical gravitational wavelength.

Exterior PM expansion ($r > d$)

We introduce the "gothic" metric

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \tag{2.4}$$

with $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. We fix de Donder gauge: $\partial_\nu \mathfrak{g}^{\mu\nu} = 0$. Einstein's equations can then be exactly written as

$$\Box \mathfrak{g}^{\mu\nu} = \frac{16\pi G}{c^4}\mathcal{T}^{\mu\nu}, \tag{2.5}$$

where $\Box = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$ and

$$\mathcal{T}^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}(\mathfrak{g}, \partial\mathfrak{g}, \partial^2\mathfrak{g}), \tag{2.6}$$

with $T^{\mu\nu}$ the stress-energy tensor and where $\Lambda^{\mu\nu} = \mathcal{O}(\mathfrak{g}^2)$ includes all non-linearities of Einstein's equations. We solve these equations by means of a post-Minkowskian expansion

$$\mathfrak{g}^{\mu\nu}_{\text{ext}} = \sum_{n=1}^{\infty} G^n h^{\mu\nu}_{(n)}, \tag{2.7}$$

where G is the Newton constant and is used to label the PM orders. We insert $\mathfrak{g}^{\mu\nu}$ into the vacuum Einstein equation:

$$G^1 \quad \to \quad \Box h^{\mu\nu}_{(1)} = 0 \quad ; \quad \partial_\nu h^{\mu\nu}_{(1)} = 0, \tag{2.8}$$

$$G^2 \quad \to \quad \Box h^{\mu\nu}_{(2)} = \Lambda^{\mu\nu}_{(2)}(h_{(1)}) \quad ; \quad \partial_\nu h^{\mu\nu}_{(2)} = 0, \tag{2.9}$$

$$\text{and so on,} \tag{2.10}$$

such that

$$\begin{cases} \Box h^{\mu\nu}_{(n)} = \Lambda^{\mu\nu}_{(n)}(h_{(1)}, \ldots, h_{(n-1)}), \\ \partial_\nu h^{\mu\nu}_{(n)} = 0. \end{cases} \tag{2.11}$$

It is interesting to combine this formalism with a multipolar expansion. This is the MPM formalism. The general monopolar solution is

$$h^{\text{mono}}(\vec{x}, t) = \frac{R\left(t - \frac{r}{c}\right) + A\left(t + \frac{r}{c}\right)}{r}, \tag{2.12}$$

which solves

$$\left(-\frac{1}{c^2}\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right) h^{\text{mono}} = 0. \tag{2.13}$$

It contains an advanced A and a retarded R solution. Since we are interested in waves produced by a localized system, we will only consider the retarded solutions. The monopolar linear solution is therefore

$$h^{\mu\nu}_{(1)} = \frac{R^{\mu\nu}(t - \frac{r}{c})}{r}. \tag{2.14}$$

Dipolar solutions are obtained by acting with ∂_i. We have

$$0 = \partial_i \Box h^{\mu\nu}_{(1)} = \Box \partial_i h^{\mu\nu}_{(1)}. \tag{2.15}$$

Therefore $\partial_i h^{\mu\nu}_{(1)}$ is a solution to the wave equation. Moreover,

$$0 = \partial_i \partial_\mu h^{\mu\nu}_{(1)} = \partial_\mu \partial_i h^{\mu\nu}_{(1)}. \tag{2.16}$$

The solution $\partial_i h^{\mu\nu}_{(1)}$ therefore obeys the harmonic gauge. Acting with one derivative increases the spherical harmonic contain by one. Hence, we found the dipolar solution

$$h^{\text{dipolar}\ \mu\nu}_{(1)} = \partial_i \left(\frac{R^{\mu\nu}_i(t - \frac{r}{c})}{r} \right). \tag{2.17}$$

where we defined one function R_i for each index i. The general multipolar solution is obtained by applying an arbitrary amount $\ell \in \mathbb{N}$ of spatial derivatives :

$$h^{\mu\nu}_{(1)}(\vec{x},t) = \sum_{\ell=0}^{\infty} \partial_L \left(\frac{R^{\mu\nu}_L(u)}{r} \right), \tag{2.18}$$

where $u = t - r/c$. Here $L = i_1 \cdots i_\ell$ is a multi-index with ℓ spatial indices and $\partial_L = \partial_{i_1 \cdots i_\ell} = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_\ell}}$. Without loss of generality we can assume that R_L is symmetric since any antisymmetric term would give an identically vanishing contribution.

We can now use the theory of representations of the $SO(3)$ group. There exists two invariant tensors under a $SO(3)$ rotation: δ_{ij} and ϵ_{ijk}. We can use these invariant tensors to perform a decomposition of $R^{\mu\nu}_L$ as a direct sum of irreducible representations under $SO(3)$ using the tensorial methods. For example, we decompose

Tensorial methods for $SO(3)$ are similar to tensorial methods for $SU(3)$ developed in the course PHYS-F485 Representations of groups and application to physics.

$$R_{ij} = \hat{R}_{ij} + \delta_{ij}\hat{R} \tag{2.19}$$

where $\hat{R} = \frac{1}{3}R_{kk}$ and $\hat{R}_{ij}$ is traceless $\hat{R}_{ii} = 0$.

There are 10 independent functions $R^{\mu\nu}_L(u)$ for each multi-index L. We now impose the harmonic condition $\partial_\nu h^{\mu\nu}_{(1)} = 0$, which gives four differential relations between the R_L's. We end up with six independent functions, labelled as six types of multipole moments. The most general solution of $\Box h^{\mu\nu}_{(1)} = 0 = \partial_\mu h^{\mu\nu}_{(1)}$ is [6]

[6] Kip S. Thorne. Multipole expansions of gravitational radiation. *Rev. Mod. Phys.*, 52:299–339, Apr 1980

$$h^{\mu\nu}_{(1)} = k^{\mu\nu}_{(1)} + \partial^\mu \varphi^\nu_{(1)} + \partial^\nu \varphi^\mu_{(1)} - \eta^{\mu\nu}\partial_\rho \varphi^\rho_{(1)}, \tag{2.20}$$

where $k^{\mu\nu}_{(1)}$ depends on two sets of STF (Symmetric Trace-Free) multipole moments: $I_L(u)$ the *mass-moment of order* ℓ and $J_L(u)$ the *current-moment of order* ℓ. The other terms correspond to a linearized gauge transformation. Moreover, $\varphi^\mu_{(1)}$ depends on four sets of moments ($\mu = 0,1,2,3$): $W_L(u), X_L(u), Y_L(u), Z_L(u)$. The analogue for electromagnetism is $E_L(u), B_L(u)$, the electric and magnetic multipolar moments of a source. Here, the explicit formulae are

We have used the multi-index notation $I_L = I_{i_1 \ldots i_\ell}$.

Dots mean derivative with respect to $u = t - \frac{r}{c}$.

$$k^{00}_{(1)} = -\frac{4}{c^2} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left(\frac{1}{r} I_L(u) \right), \tag{2.21}$$

$$k^{0i}_{(1)} = \frac{4}{c^3} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell!} \left[\partial_{L-1} \left(\frac{1}{r} \dot{I}_{iL-1}(u) \right) + \frac{\ell}{\ell+1} \varepsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} J_{bL-1}(u) \right) \right], \tag{2.22}$$

$$k^{ij}_{(1)} = -\frac{4}{c^4} \sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell!} \left[\partial_{L-2} \left(\frac{1}{r} \ddot{I}_{ijL-2}(u) \right) + \frac{2\ell}{\ell+1} \partial_{aL-2} \left(\frac{1}{r} \varepsilon_{ab(i} \dot{J}_{j)L-2} \right) \right], \tag{2.23}$$

where $I_L(u)$ and $J_L(u)$, denoted as the source moments, are arbitrary functions of retarded time u except for the lowest ℓ cases as a consequence of Poincaré conservation laws:

$M = I =$ constant	Total mass	Noether charge of ∂_t
$X_i = \frac{I_i}{I} =$ constant	Center-of-mass position	Noether charge of boosts
$P_i = \dot{I}_i =$ constant	Linear momentum	Noether charge of ∂_i
$S_i = J_i =$ constant	Angular momentum	Noether charge of rotations

Such conservation laws are here technically enforced due to the harmonicity condition. In linear theory, these conservation laws are exact. In the non-linear theory such conservation laws become flux-balance laws which depend upon the emission of gravitational waves.

Near-zone PN expansion of a 2-body system

We consider two bodies with position $\vec{z}_i(t)$, velocity $\vec{v}_i(t)$ and spin $\vec{s}_i(t)$, where $i = A, B$. There could be higher multipolar moments. In that case, one would need the Mathisson-Papapetrou-Dixon theory, which goes beyond this class. This is the *skeletonization* of compact bodies: we approximate their structure in an effective manner from their lower order multipolar moments.

In Newtonian theory, gravity arises from a potential $U(t, \vec{x})$ that obeys Gauss's law:

$$\Delta U(t, \vec{x}) = 4\pi G \rho(t, \vec{x}), \tag{2.24}$$

where ρ is the energy density. For two point particles, it is given by

$$\rho = \sum_{A,B} m_A \delta^{(3)}(\vec{x} - \vec{z}_A(t)). \tag{2.25}$$

Using $\Delta\left(1/|\vec{x}|\right) = -4\pi\delta^{(3)}(\vec{x})$, we have

$$U(t, \vec{x}) = -\sum_{A,B} \frac{G m_A}{|\vec{x} - \vec{z}_A(t)|}. \tag{2.26}$$

This field is singular at the locations of the particles. Two regularization schemes exist: Hadamard and dimensional regularization. The latter regularization is more powerful since it removes most of the possible ambiguities. We are not going to explain these regularization schemes here. Let us set $G = 1$ from now on in this section. After regularization, one obtains

$$U(t, \vec{x} = \vec{z}_A(t)) = -\frac{m_B}{|\vec{z}_A - \vec{z}_B|}. \tag{2.27}$$

The action is then given by

$$\mathcal{S} = \int dt \left\{ \frac{1}{2} \sum_{A,B} m_A \vec{v}_A^2 + \frac{m_A m_B}{|\vec{z}_A - \vec{z}_B|} \right\}, \tag{2.28}$$

the first term being the kinetic part and the second the potential. The 2-body Newtonian Lagrangian is $\mathcal{S} = \int dt\, L_N$. The Euler-Lagrange equations are

$$\frac{d\vec{v}_A}{dt} = -m_B \frac{\vec{R}}{|\vec{R}|^3} \quad ; \quad \frac{d\vec{v}_B}{dt} = m_A \frac{\vec{R}}{|\vec{R}|^3}, \tag{2.29}$$

where $\vec{R} = \vec{z}_A - \vec{z}_B$. In the center of mass frame, $m_A \vec{z}_A + m_B \vec{z}_B = 0$. We can reach it using translation. The remaining equation is

$$\frac{d^2\vec{R}}{dt^2} = -M \frac{\vec{R}}{|\vec{R}|^3} \quad ; \quad M = m_A + m_B. \tag{2.30}$$

We go to polar coordinates (R, ϕ) where $\vec{R} = R\vec{e}_R$. We have $\dot{e}_R = \dot{\phi} e_R$ and $\dot{e}_\phi = -\dot{\phi} e_R$, such that

$$\frac{d^2\vec{R}}{dt^2} = \left(\ddot{R} - R\dot{\phi}^2\right)\vec{e}_R + \left(R\ddot{\phi} + 2\dot{R}\dot{\phi}\right)\vec{e}_\phi. \tag{2.31}$$

Comparing with the former equation, we have

$$\ddot{R} - R\dot{\phi}^2 = -\frac{M}{R^2} \quad ; \quad R\ddot{\phi} + 2\dot{R}\dot{\phi} = 0. \tag{2.32}$$

The second equation gives $R^2\dot{\phi} = L$ constant, which is the angular momentum. Inserting in the first one and integrating yields

$$\left(\frac{dR}{dt}\right)^2 = 2E + \frac{2M}{R} - \frac{L^2}{R^2}, \tag{2.33}$$

where the energy E is a second constant of motion. Using the chain rule, we have $\frac{dR}{dt} = \frac{dR}{d\phi}\frac{d\phi}{dt}$. By playing with equations, one gets

$$\frac{d^2}{d\phi^2}\left(\frac{1}{R}\right) + \frac{1}{R} = \frac{M}{L^2}, \tag{2.34}$$

The solution to this equation is given by the Kepler ellipses of semi-latus rectum p and eccentricity e given by

$$R(\phi) = \frac{p}{1 + e\cos(\phi - \phi_0)}, \tag{2.35}$$

where $\begin{cases} p = L^2/M, \\ e = \sqrt{1 + 2EL^2/M^2}. \end{cases}$

Metric at 1PN order

We will aim to solve for the dynamics at 1PN order, which will allow to introduce already many techniques. Let us first formulate an ansatz for the metric:

$$\begin{cases} g_{tt} = -e^{2U} + \mathcal{O}\left(\left(v/c\right)^6\right), \\ g_{ti} = 4g_i + \mathcal{O}\left(\left(v/c\right)^5\right), \\ g_{ij} = \delta_{ij}e^{-2U} + \mathcal{O}\left(\left(v/c\right)^4\right), \end{cases} \tag{2.36}$$

Here the Newtonian potential U is of order 2 in the v/c expansion and g_i is of order 3. The Newtonian order is such that the geodesic motion of a body in the field of other body produces the Newtonian equations of motion. Let us derive Einstein's equations. We first compute the inverse metric in the perturbative expansion, which is straightforward. Second,we compute the Christoffel symbols. For example,

$$\begin{aligned} \Gamma^t_{tt} &= \frac{1}{2}g^{t\sigma}\left(\partial_t g_{t\sigma} + \partial_t g_{\sigma t} - \partial_\sigma g_{tt}\right) \\ &= \frac{1}{2}g^{tt}\partial_t g_{tt} + \frac{1}{2}g^{ti}\left(2\partial_t g_{ti} - \partial_i g_{tt}\right) \\ &\frac{1}{2}(-e^{-2U})(-2e^{2U}\dot{U}) + O(v^5) &(2.37) \\ &= \dot{U} + \mathcal{O}(v^5), &(2.38) \end{aligned}$$

The computation of the other Christoffel symbols follows similarly. We can then compute the Ricci tensor, and we obtain the two relevant components:

$$\begin{cases} R^{tt} = \Delta U + \partial_t\left(3\dot{U} + 4\partial_i g^i\right) + \mathcal{O}(v^5), \\ R^{ti} = 2\Delta g^i - 2\partial^i\left(\dot{U} + \partial_j g^j\right) + \mathcal{O}(v^5). \end{cases} \tag{2.39}$$

We now fix the gauge. Two popular gauges can be used:

• Harmonic gauge:

Note that the harmonic gauge condition is sometimes called *de Donder* gauge condition, which is different from the one seen in Eq. (1.29).

$$\Box X^\mu = 0 \quad \Leftrightarrow \quad \partial_\nu\left(\sqrt{-g}g^{\nu\alpha}\partial_\alpha X^\mu\right) = 0. \tag{2.40}$$

In our case, this implies

$$\dot{U} + \partial_i g^i = 0. \tag{2.41}$$

• Coulomb gauge:

$$\Delta X^\mu = 0, \tag{2.42}$$

which implies

$$3\dot{U} + 4\partial_i g^i = 0. \tag{2.43}$$

Let us focus on Coulomb gauge. Then, the Ricci tensor components become

$$\begin{cases} R^{tt}|_{\text{Coul.}} = \Delta U + \mathcal{O}(v^5), \\ R^{ti}|_{\text{Coul.}} = 2\Delta\xi^i + \mathcal{O}(v^5), \end{cases} \tag{2.44}$$

where $\xi^i = g^i - \frac{1}{4}\partial^i\dot{\chi}$, and $\Delta\chi = U$. We focus on the right-hand side of the Einstein's field equations, which corresponds to the stress-energy tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_m}{\delta g_{\mu\nu}}, \tag{2.45}$$

with

$$\begin{aligned}\mathcal{S}_m &= -\sum_{A,B} m_A \int d\tau_A \\ &= -\sum_{A,B} m_A \int d^4x\, \delta^{(4)}(x^\lambda - z_A^\lambda)\sqrt{-\,g_{\mu\nu}\big|_A\, dz_A^\mu dz_A^\nu}.\end{aligned} \tag{2.46}$$

We obtain

$$T^{\mu\nu} = \sum_{A,B} m_A \int d\tau_A\, \delta^{(4)}(x^\lambda - z_A^\lambda)\frac{u_A^\mu u_A^\nu}{\sqrt{-g}\sqrt{-g_{\mu\nu}u_A^\mu u_B^\nu}}, \tag{2.47}$$

where $u_A^\mu = \frac{dt}{d\tau_A}v_A^\mu$ and $v_A^\mu = (v_A^t, v_A^i) = \left(1, \frac{d\vec{z}_A}{dt}\right)$ such that

$$T^{\mu\nu} = \sum_{A,B} m_A \delta^{(3)}(\vec{x} - \vec{z}_A(t))\frac{v_A^\mu v_A^\nu}{\sqrt{g g_{\alpha\beta} v_A^\alpha v_B^\beta}}. \tag{2.48}$$

Moreover, the denominator reads as

$$\begin{aligned} g g_{\alpha\beta} v^\alpha v^\beta &= g\left(g_{tt} + 2g_{ti}v^i + g_{ij}v^i v^j\right) \\ &= 1 - 2U - v^2 + \mathcal{O}(v^4).\end{aligned} \tag{2.49}$$

This yields

$$T^{\mu\nu}(t,\vec{x}) = \sum_{A,B} m_A \delta^{(3)}\left(\vec{x} - \vec{z}_A(t)\right) v_A^\mu v_A^\nu \left(1 + U + \frac{v^2}{2}\right) + \mathcal{O}(v^4), \tag{2.50}$$

where $v_A^\mu = (1, v_A^i)$. Finally, Einstein's equations can be recast as $R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)$ and we find

$$\begin{cases} R^{tt} = 4\pi\left(T^{tt} + T^{ii}\right) + \mathcal{O}(v^4), \\ R^{ti} = 8\pi T^{ti} + \mathcal{O}(v^3). \end{cases} \tag{2.51}$$

The final equations, up to lower PN corrections, are

$$\boxed{\begin{aligned} \Delta U + \partial_t\left(3\dot{U} + 4\partial_i g^i\right) &= 4\pi \sum_{A,B} m_A \delta^{(3)}(\vec{x} - \vec{z}_A(t))\left(1 + U + \frac{3}{2}v_A^2\right), \\ \Delta g^i - \partial^i\left(\dot{U} + \partial_j g^j\right) &= 4\pi \sum_{A,B} m_A \delta^{(3)}(\vec{x} - \vec{z}_A(t)) v_A^i. \end{aligned}}$$

Solution in the Coulomb gauge (CG)

In the Coulomb gauge, because the condition (2.43), the first equation becomes

$$\Delta U_{\mathrm{CG}} = 4\pi \sum_{A,B} m_A \delta^{(3)}(\vec{x} - \vec{z}_A(t)) \left(1 + U_{\mathrm{CG}} + \frac{3}{2} v_A^2\right). \tag{2.53}$$

We solve it iteratively, by injecting the 0PN solution on the source term, solving at 1PN the equation and so forth. At 0PN, we have

$$U_{\mathrm{CG}}^{(0)}(t, \vec{x}) = -\sum_{A,B} \frac{m_A}{|\vec{x} - \vec{z}_A(t)|}. \tag{2.54}$$

After a regularization, using Hadamard or dimensional one, we neglect the divergent terms and obtain

$$U_{\mathrm{CG}}^{(0)}(t, \vec{z}_A(t)) = -\frac{m_A}{|\vec{z}_B(t) - \vec{z}_A(t)|}. \tag{2.55}$$

We deduce

$$\Delta U_{\mathrm{CG}}^{(2)} = 4\pi \sum_{A,B} m_A \delta^{(3)}(\vec{x} - \vec{z}_A(t)) \left(1 + \frac{3}{2} v_A^2 - \frac{m_B}{|\vec{z}_A(t) - \vec{z}_B(t)|}\right). \tag{2.56}$$

We then obtain

$$U_{\mathrm{CG}} = U_{\mathrm{CG}}^{(0)} + U_{\mathrm{CG}}^{(2)} = -\sum_{A,B} \frac{m_A}{|\vec{x} - \vec{z}_A(t)|} \left(1 + \frac{3}{2} v_A^2 - \frac{m_B}{|\vec{z}_A(t) - \vec{z}_B(t)|}\right). \tag{2.57}$$

The second equation is

$$\Delta g_{\mathrm{CG}}^i - \frac{1}{4} \partial^i \dot{U}_{\mathrm{CG}} = 4\pi \sum_{A,B} m_A \delta^{(3)}(\vec{x} - \vec{z}_A(t)) v_A^i. \tag{2.58}$$

Now, remember that $\vec{g} = \vec{\xi} + \frac{1}{4} \vec{\nabla} \dot{\chi}$, where $\Delta \chi = U$. Then,

$$\Delta \vec{\xi} = 4\pi \sum_{A,B} \delta^{(3)}(\vec{x} - \vec{z}_A(t)) \vec{v}_A. \tag{2.59}$$

A solution of this equation is given by

$$\vec{\xi} = -\sum_{A,B} \frac{m_A \vec{v}_A}{|\vec{x} - \vec{z}_A(t)|}. \tag{2.60}$$

We want to solve

$$\Delta \chi_{\mathrm{CG}} = U = -\sum_{A,B} \frac{m_A}{|\vec{x} - \vec{z}_A(t)|} \tag{2.61}$$

at 0PN because we will take a time derivative of χ, which will increase the PN order by one. We find, using $\Delta |\vec{r}| = 2/|\vec{r}|$, that

$$\chi_{\mathrm{CG}} = -\frac{1}{2} \sum_{A,B} m_A |\vec{x} - \vec{z}_A(t)|. \tag{2.62}$$

Then, this implies that

$$\nabla \dot{\chi}_{\rm CG} = -\frac{1}{2} \sum_{A,B} \frac{m_A}{|\vec{x} - \vec{z}_A(t)|} \left(\vec{v}_A - (\vec{n}_A \cdot \vec{v}_A)\vec{n}_A \right), \tag{2.63}$$

where $\vec{n}_A = \vec{x} - \frac{\vec{z}_A}{|\vec{x}-\vec{z}_A|}$. Finally, the solution at 1PN is

$$\vec{g}_{\rm CG}(t, \vec{x}) = -\frac{1}{8} \sum_{A,B} \frac{m_A}{|\vec{x} - \vec{z}_A(t)|} \left(7\vec{v}_A + (\vec{n}_A \cdot \vec{v}_A)\vec{n}_A \right) + \mathcal{O}(v^5). \tag{2.64}$$

Now that we have the field, we can find the 1PN two-body Lagrangian. At low PN orders, we can use the *Fichtenholz trick*. We can find the Lagrangian of the body A in the field of the bodies A and B. Then, we symmetrize:

$$L_{AB} = \left(L_A[g_{\mu\nu}|_{A/B}] + L_B[g_{\mu\nu}|_{A/B}] \right). \tag{2.65}$$

The Lagrangian of body A in the field of bodies A/B is

$$L_A = -m_A \sqrt{-g_{\mu\nu}|_{A/B} \frac{dz_A^\mu}{dt} \frac{dz_A^\nu}{dt}}, \tag{2.66}$$

such that its action is given by

$$\mathcal{S}_A = -m_A \int d\tau_A = \int dt\, L_A. \tag{2.67}$$

To develop this Lagrangian, we use the two-body metric that we found

$$ds^2 = -e^{2U} dt^2 + 4 g_i dt dx^i + e^{-2U} d\vec{x}^2, \tag{2.68}$$

and we have

$$\begin{aligned} L_A &= -m_A e^U \left(1 - v_A^2 e^{-4U} - 8 g_i v_A^i e^{-2U} \right)^{1/2} \\ &= -m_A \left(1 - \frac{v_A^2}{2} + U - \frac{1}{8} v_A^4 + \frac{3}{2} v_A^2 U + \frac{U^2}{2} - 4 g_i v^i \right) + \mathcal{O}(v^6). \end{aligned} \tag{2.69}$$

We now need to evaluate that Lagrangian precisely at the location where $\vec{x} = \vec{z}_A(t)$, where all fields admit divergences. After regularization and in the Coulomb gauge, we find

$$\begin{aligned} L_A^{\rm CG} = &- m_1 + \frac{1}{2} m_A v_A^2 + \frac{m_A m_B}{R} + \frac{1}{8} m_A v_A^4 \\ &+ \frac{m_A m_B}{2R} \left[3(v_A^2 + v_B^2) - 7\vec{v}_A \cdot \vec{v}_B - (\vec{N} \cdot \vec{v}_A)(\vec{N} \cdot \vec{v}_B) \right] \\ &- \frac{m_A m_B^2}{R^2} + \mathcal{O}(v^6), \end{aligned} \tag{2.70}$$

where $\vec{z}_A - \vec{z}_B = R\vec{N}$ and $\vec{N} \cdot \vec{N} = 1$. Finally, we symmetrize to obtain the Fichtenholz Lagrangian at 1PN, in the Coulomb gauge. We obtain

$$
\boxed{\begin{aligned}
L^{\text{CG}}_{AB} =& -(m_A + m_B) + \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 + \frac{m_A m_B}{R} \\
&+ \frac{1}{8} m_A v_A^4 + \frac{1}{8} m_B v_B^4 \\
&+ \frac{m_A m_B}{2R} \left[3(v_A^2 + v_B^2) - 7\vec{v}_A \cdot \vec{v}_B - (\vec{N} \cdot \vec{v}_A)(\vec{N} \cdot \vec{v}_B) \right] \\
&- \frac{m_A m_B (m_A + m_B)}{2R^2} + \mathcal{O}(v^6),
\end{aligned}}
$$

where the first four terms correspond to the 0PN order and the following ones to the 1PN order. The resulting equations of motion are the so-called *Einstein-Infeld-Hoffmann* or *EIH* equations derived in 1938. The derivation of these equations is left for the exercise sessions. More generally, one can derive the 4PN Lagrangian that describes the conservative part of the dynamics using the so-called *Fokker Lagrangian*.

Qualitatively, new features arise at 2.5PN order. The conservative Hamiltonian/Lagrangian equations of motion need to be complemented with the dissipative part of the dynamics: the radiation-reaction of the bodies due to the emission of gravitational waves. The first description of the dissipative effects is due to Infeld and Plebonski in 1960. We will not consider this formalism here.

Let us construct the Fokker Lagrangian. We start with the action of gravity plus the matter fields:

$$\mathcal{S}[g_{\mu\nu}, \psi] = \mathcal{S}_{\text{EH}}[g_{\mu\nu}] + \mathcal{S}_m[g_{\mu\nu}, \psi] + \text{boundary terms} + \text{gauge fixing terms}. \tag{2.72}$$

The ψ dependence encodes the two bodies, $\mathcal{S}_{\text{EH}}$ is the Einstein-Hilbert action, the boundary terms do not change the dynamics and the gauge fixing terms are included to remove part of diffeomorphism covariance. We will gauge fix to harmonic coordinates. It is convenient to work with the first order Lagrangian such that

Since Einstein's equations are second order in derivatives, a first order Lagrangian has to exist. It is just not generally covariant.

$$
\begin{aligned}
&\mathcal{S}_{\text{EH}}[g] + \text{bnd. terms} + \text{g.f. terms} \\
&= \frac{1}{16\pi G} \int \sqrt{-g} \left\{ g^{\mu\nu} \left(\Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\lambda} \right) - \frac{1}{2} g_{\mu\nu} \Gamma^\mu \Gamma^\nu \right\},
\end{aligned} \tag{2.73}
$$

where $\Gamma^\mu = g^{\nu\rho} \Gamma^\mu_{\nu\rho}$. The matter action is the *Dixon action* for each body but taking only into account the mass of the body (the lowest multipole: the monopole), we have

$$\mathcal{S}_m[g_{\mu\nu}, \psi] = -\sum_{A,B} m_A \int dt \sqrt{- g_{\mu\nu}\big|_{x^A} \frac{dx^\mu_A}{dt} \frac{dx^\nu_A}{dt}}. \tag{2.74}$$

The Fokker action is now defined as

$$\mathcal{S}_F[\psi] = \mathcal{S}_{\text{EH}} \left[\bar{g}_{\mu\nu}[\psi] \right] + \mathcal{S}_m \left[\bar{g}_{\mu\nu}[\psi], \psi \right], \tag{2.75}$$

where $\bar{g}_{\mu\nu}[\psi]$ is a solution of Einstein's equations $\frac{\delta \mathcal{S}_{\text{tot}}}{\delta g_{\mu\nu}} = 0$ to the desired PN order. In the case of point particles,

$$\mathcal{S}_F[\psi] = \int dt\, L_F[z_A(t), z_B(t), v_A(t), v_B(t), a_A(t), a_B(t)], \tag{2.76}$$

where a_i is the acceleration. We note that the transformation

$$L_F \to L_F + \frac{d}{dt} B[z_A(t), z_B(t), v_A(t), v_B(t), a_A(t), a_B(t)] \tag{2.77}$$

does not change the equations of motion. Moreover, we can change coordinates as

$$\begin{cases} z'_A = \delta z_A + z_A \\ z'_B = \delta z_B + z_B, \end{cases} \tag{2.78}$$

which will modify the Lagrangian. As proven in [7] we can use these two ambiguities to find a Fokker Lagrangian that does not depend upon accelerations at least up to 4PN order. The equations of motion are

$$\frac{\delta L_F}{\delta z^i_A} = \frac{\partial L_F}{\partial z^i_A} - \frac{d}{dt}\frac{\partial L_F}{\partial v^i_A} = F_i^{\text{react}}, \tag{2.79}$$

with F_i^{react} the radiation reaction force. At 2.5PN order, in harmonic coordinates, it constitutes the dominant term:

$$F_i^{\text{react}}\big|_{\text{harm}} = \frac{G}{c^5}\rho \left[\frac{3}{5} x^j Q_{ij}^{(5)} + 2\frac{d}{dt}\left(v^j Q_{ij}^{(3)}\right) - Q_{jk}^{(3)} \partial_{ijk}\chi\right] + \mathcal{O}(1/c^7), \tag{2.80}$$

where ρ is the density: $\rho = \sum_{A,B} m_A \delta^{(3)}(\vec{x} - \vec{z}_A(t))$,
$Q_{ij} = m_A x^i_A x^j_A + m_B x^i_B x^j_B$ is the quadrupole moment [8] and

$$\begin{cases} \Delta U = -4\pi G \rho, \\ \Delta \chi = 2U. \end{cases} \tag{2.81}$$

[7] Laura Bernard, Luc Blanchet, Alejandro Bohé, Guillaume Faye, and Sylvain Marsat. Fokker action of nonspinning compact binaries at the fourth post-newtonian approximation. *Physical Review D*, 93(8), April 2016

[8] S. Chandrasekhar and F. Paul Esposito. The 2½-post-newtonian equations of hydrodynamics and radiation reaction in general relativity. 160:153, 1970

2.2 *Effective One-Body methods*

We know that the two-body problem in Newtonian theory can be effectively described as a one-body problem. It turns out that the same is true in General Relativity. The resulting dynamical description is called the effective one-body (EOB) method. However, because of non-linearities, the dynamics is not known exactly but it can be deduced perturbatively in different approximation schemes (PN/PM, self-force, numerical relativity). We will now look at the effective one-body method, in the PN validity regime [9]. At low PN orders, it is sufficient to find a Hamiltonian since all dynamics is conservative. We will derive the Hamiltonian at 1PN order. We will proceed through the following steps:

(i) Write the Hamiltonian of the relative motion of the two-body problem in polar coordinates. It will have the form

$$H_{1\text{PN}} = H[Q, P], \tag{2.82}$$

[9] A. Buonanno and T. Damour. Effective one-body approach to general relativistic two-body dynamics. *Physical Review D*, 59(8), March 1999; and Nathalie Deruelle and Jean-Philippe Uzan. *Relativity in Modern Physics*. Oxford Graduate Texts. Oxford University Press, 8 2018

with $Q = (R, \Phi)$ are the relative positions and $P = (P_R, P_\Phi)$ are the conjugate relative momenta. The relative position will be parametrized in the plane of motion $\vec{x}_2 - \vec{x}_1 = (R\cos\Phi, R\sin\Phi)$. The radial and polar momenta are related to the Cartesian components of the momenta as

$$P_R = (\vec{n}_2 - \vec{n}_1)\cdot\vec{P} \quad ; \quad P_\Phi = R|(\vec{n}_2 - \vec{n}_1)\times\vec{P}|, \tag{2.83}$$

with $\vec{n}_i = \frac{\vec{x}_i}{R}$.

(ii) Write the Hamiltonian of a test particle of mass equal to the reduced mass $\mu = m_1 m_2/M$, where $M = m_1 + m_2$ is the total mass, in a Schwarzschild-like geometry of mass M:

$$H_e[q, p], \tag{2.84}$$

where $q = (r, \phi)$ and $p = (p_r, p_\phi)$ are the dynamic variables of the planar motion in the Schwarzschild-like geometry: the positions of the test particle and their conjugate momenta.

(iii) Introduce a generating function $F(q, Q)$ such that

$$dF = P_R dR + P_\Phi d\Phi - \left(p_r dr + p_\phi d\phi\right). \tag{2.85}$$

It defines a canonical transformation from (q, P) to (Q, p).

(iv) Fix $F(q, Q)$ in order to find a functional relation between $H_{1\mathrm{PN}}[Q, P]$ and $H_{\mathrm{EOB}}[q, p]$, when rewritten with the same variables (Q, p).

We shall now follow this program step by step.

(i) Hamiltonian of the relative motion

The 1PN Lagrangian is (we are setting $c = 1$)

$$\begin{aligned} L_{1\mathrm{PN}} = &-(m_1 + m_2) + \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 + \frac{m_1 m_2}{R} + \frac{1}{8}m_1\left(\vec{v}_1^2\right)^2 + \frac{1}{8}m_2\left(\vec{v}_2^2\right)^2 \\ &+ \frac{m_1 m_2}{2R}\left[3(\vec{v}_1^2 + \vec{v}_2^2) - 7\vec{v}_1\cdot\vec{v}_2 - \left(\vec{N}\cdot\vec{v}_1\right)\left(\vec{N}\cdot\vec{v}_2\right)\right] - \frac{m_1 m_2(m_1 + m_2)}{2R^2}, \end{aligned} \tag{2.86}$$

where $\vec{x}_2 - \vec{x}_1 = R\vec{N}$ and $\vec{N} = (\cos\Phi, \sin\Phi)$. We could go to the center-of-mass frame where $m_1\vec{v}_1 + m_2\vec{v}_2 = 0$, but it is not necessary yet. We just switch to the Hamiltonian formulation

$$\begin{cases} \vec{P}_1 = \dfrac{\partial L_{1PN}}{\partial \vec{v}_1} = m_1\vec{v}_1 + \frac{1}{2}m_1(\vec{v}_1^2)\vec{v}_1 + \dots \\ \vec{P}_2 = \dfrac{\partial L_{1PN}}{\partial \vec{v}_2} = m_1\vec{v}_2 + \frac{1}{2}m_2(\vec{v}_2^2)\vec{v}_2 + \dots \end{cases} \tag{2.87}$$

At 1PN order, we can invert the relation such that

$$\vec{v}_1 = \frac{\vec{P}_1}{m_1} - \frac{\vec{P}_1^2}{2m_1^3}\vec{P}_1 + O(v^5/c^5). \tag{2.88}$$

The 1PN Hamiltonian is thus

$$H_{\rm 1PN} = \sum_{1,2} \vec{P}_i \cdot \vec{v}_i - L_{1PN}, \tag{2.89}$$

where all $\vec{v}_i$ are replaced by $\vec{P}_i$. We get

$$\begin{aligned} H_{\rm 1PN} =& m_1 + m_2 + \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} - \frac{m_1 m_2}{R} - \frac{P_1^4}{8m_1^3} - \frac{P_2^4}{8m_2^3} \\ &- \frac{m_1 m_2}{2R}\left[3\left(\frac{P_1^2}{m_1^2} + \frac{P_2^2}{m_2^2}\right) - 7\frac{\vec{P}_1 \cdot \vec{P}_2}{m_1 m_2} - \frac{(\vec{N}\cdot\vec{P}_1)(\vec{N}\cdot\vec{P}_2)}{m_1 m_2}\right] \\ &+ \frac{m_1 m_2 (m_1 + m_2)}{2R^2}. \end{aligned} \tag{2.90}$$

Introducing again $M = m_1 + m_2$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$, $\nu = \frac{\mu}{M}$, going to the center-of-mass frame, where $\vec{P}_1 = -\vec{P}_2 = \vec{P}$, and introducing the dimensionless quantities $\vec{\hat{P}} = \frac{\vec{P}}{\mu}$, $\hat{R} = \frac{R}{M}$, the Hamiltonian becomes

$$\frac{H_{\rm 1PN} - M}{\mu} = \frac{\hat{P}^2}{2} - \frac{1}{\hat{R}} - \frac{1-3\nu}{8}\hat{P}^4 - (3+\nu)\frac{\hat{P}^2}{2\hat{R}} - \frac{\nu}{2\hat{R}}\left(\vec{N}\cdot\vec{\hat{P}}\right) + \frac{1}{2\hat{R}^2}. \tag{2.91}$$

We switch to polar coordinates and we have

$$\vec{\hat{R}} = (\hat{R}\cos\Phi, \hat{R}\sin\Phi) \quad ; \quad \vec{\hat{P}} = (\vec{N}\cdot\vec{\hat{P}}, |\vec{N}\times\vec{\hat{P}}|). \tag{2.92}$$

Then, we find the conjugated momenta corresponding to R and Φ:

$$\hat{P}_R = \vec{N}\cdot\vec{\hat{P}} \quad ; \quad \hat{P}_\Phi = |\vec{N}\times\vec{\hat{P}}|\hat{R}, \tag{2.93}$$

such that

$$\hat{P}^2 = \hat{P}_R^2 + \frac{\hat{P}_\Phi^2}{\hat{R}^2} \quad ; \quad \left(\vec{N}\cdot\vec{\hat{P}}\right) = \hat{P}_R. \tag{2.94}$$

Therefore, by injecting those results in the expression of $H_{\rm 1PN}$, we finally obtain

$$\boxed{\frac{H_{\rm 1PN}[Q,P] - M}{\mu} = \frac{\hat{P}^2}{2} - \frac{1}{\hat{R}} - \frac{1-3\nu}{8}\hat{P}^4 - (3+\nu)\frac{\hat{P}^2}{2\hat{R}} - \frac{\nu}{2\hat{R}}\hat{P}_R + \frac{1}{2\hat{R}^2},} \tag{2.95}$$

in the set of coordinates $[Q,P] = [R, \Phi, P_R, P_\Phi]$.

(ii) Hamiltonian of a test particle

We now consider a test particle in a static and spherically symmetric metric. The most general such metric can be written as

In more generality we could consider for the higher PN order a stationary and axisymmetric metric instead of a static and spherically symmetric metric. In that case, any Ricci-flat metric can be written in coordinates such that it contains only one non-diagonal term, thus, four arbitrary functions after fixing the radius such that $g_{\phi\phi} = r^2$: $g_{tt}, g_{\theta\theta}, g_{rr}$ and $g_{t\phi}$.

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2, \tag{2.96}$$

on the $\theta = \pi/2$ plane. The action for a test particle of mass μ at coordinates (r, ϕ) is

$$\mathcal{S}_e = \int dt\, L_e, \tag{2.97}$$

where the Lagrangian is

$$L_e = -\mu\sqrt{A(r) - B(r)\dot{r}^2 - r^2\dot{\phi}^2}. \tag{2.98}$$

The conjugate momenta are

$$p_r = \frac{\partial L_e}{\partial \dot{r}} \quad ; \quad p_\phi = \frac{\partial L_e}{\partial \dot{\phi}}, \tag{2.99}$$

and the Hamiltonian is given by

$$H_e = p_r\dot{r} + p_\phi\dot{\phi} - L_e. \tag{2.100}$$

We introduce the dimensionless quantities

$$\hat{r} = \frac{r}{M} \quad , \quad \hat{p}_r = \frac{p_r}{\mu} \quad \text{and} \quad \hat{p}_\phi = \frac{p_\phi}{\mu M}. \tag{2.101}$$

We have that $\hat{p}^2 = \hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{r}^2}$. After performing some algebra, we find the Hamiltonian

$$\frac{H_e}{\mu} = \sqrt{A\left(1 + \frac{\hat{p}_r^2}{B} + \frac{\hat{p}_\phi^2}{\hat{r}^2}\right)}. \tag{2.102}$$

At 1PN order, we consider $v^2/c^2 \sim 1/\hat{r}$ corrections to the PN potential and

$$\begin{cases} A(r) = 1 + \dfrac{a_1}{\hat{r}} + \dfrac{a_2}{\hat{r}^2}, \\ B(r) = 1 + \dfrac{b_1}{\hat{r}}. \end{cases} \tag{2.103}$$

We call $q = (r, \phi)$ and $p = (p_r, p_\phi)$. Finally, after some algebra, we obtain

$$\frac{H_e[p,q]}{\mu} - 1 = \left(\frac{\hat{p}^2}{2} + \frac{a_1}{2\hat{r}}\right) - \frac{\hat{p}^4}{8} + a_1\frac{\hat{p}^2}{4\hat{r}^2} - b_1\frac{\hat{p}_r^2}{2\hat{r}} + \frac{a_2 - a_1^2/4}{2\hat{r}^2} + \mathcal{O}(v^6/c^6). \tag{2.104}$$

The EOB Hamiltonian will be a functional of H_e, as we will see below.

(iii) Generating function

We are now ready to map the 1PN Hamiltonian to the Hamiltonian describing the motion of a test particle in a stationary spherically symmetric metric. We define the canonical transformation $F(q;Q)$ such that

$$dF = P_R dR + P_\Phi d\Phi - \left(p_r dr + p_\phi d\phi\right). \tag{2.105}$$

Here q corresponds to the effective one-body coordinates and Q to the two-body coordinates. Instead of $F(q;Q)$, in order to obtain the simple equation (2.109) below, we can define $G(Q;p)$ as

$$G(Q;p) = F(q;Q) + \left(p_r r + p_\phi \phi\right) - \left(p_r R + p_\phi \Phi\right), \tag{2.106}$$

such that

$$dG = (P_R - p_r)\, dR + \left(P_\Phi - p_\phi\right) d\Phi + (r - R)\, dp_r + (\phi - \Phi)\, dp_\phi. \tag{2.107}$$

Given that $G(Q;p) = G(R,\Phi;p_r,p_\phi)$, we obviously have

$$dG = \frac{\partial G}{\partial R} dR + \frac{\partial G}{\partial \Phi} d\Phi + \frac{\partial G}{\partial p_r} dp_r + \frac{\partial G}{\partial p_\phi} dp_\phi. \tag{2.108}$$

Comparing Eqs. (2.107) and (2.108), we obtain

$$r = R + \frac{\partial G}{\partial p_r} \quad , \quad \phi = \Phi + \frac{\partial G}{\partial p_\phi} \quad , \quad P_R = p_r + \frac{\partial G}{\partial R} \quad , \quad P_\Phi = p_\phi + \frac{\partial G}{\partial \Phi}. \tag{2.109}$$

We need to investigate whether such a canonical transformation $G(Q;p)$ exists, which is built from dimensionless quantities up to an overall scaling $\sim \mu M$ that are at most 1PN. We consider the following ansatz:

$$\frac{G(Q,p)}{\mu M} = \hat{R}\hat{p}_r \left(\alpha_1 \left(\hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{R}^2}\right) + \beta_1 \hat{p}_r^2 + \frac{\gamma_1}{\hat{R}}\right), \tag{2.110}$$

where $\hat{p}_r = \frac{p_r}{\mu}, \hat{p}_\phi = \frac{p_\phi}{\mu M}, \hat{r} = \frac{r}{M}$ and $\hat{R} = \frac{R}{M}$. The ansatz gives the following relations:

$$\hat{P}_R = \hat{p}_r \left(1 + (2\alpha_1 + \beta_1)\, \hat{p}_r^2 - \alpha_1 \left(\hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{R}^2}\right)\right), \tag{2.111}$$

$$\hat{P}_\Phi = \hat{p}_\phi, \tag{2.112}$$

$$\phi = \Phi + 2\alpha_1 \frac{\hat{p}_r \hat{p}_\phi}{\hat{R}}, \tag{2.113}$$

$$\hat{r} = \hat{R} \left(1 + \alpha_1 \left(\hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{R}^2}\right) + (2\alpha_1 + 3\beta_1)\, \hat{p}_r^2 + \frac{\gamma_1}{\hat{R}}\right). \tag{2.114}$$

In order to compare both Hamiltonian that we have derived, we need to rewrite them in terms of the same variables. Recall that we have obtained

$$\frac{H_{1\mathrm{PN}}[Q,P] - M}{\mu} = \left(\frac{\hat{P}^2}{2} - \frac{h_N}{\hat{R}}\right) + h_1 \hat{P}^4 + h_2 \hat{P}^2 \hat{P}_R^2 + h_3 \hat{P}_R^4 + h_4 \frac{\hat{P}^2}{\hat{R}} + h_5 \frac{\hat{P}_R^2}{\hat{R}} + \frac{h_6}{\hat{R}^2}, \tag{2.115}$$

where the coefficients h_i are given by

$$h_N = 1 \quad ; \quad h_1 = -\frac{1-3\nu}{8} \quad ; \quad h_2 = 0 = h_3 \quad ;$$
$$h_4 = -\frac{3+\nu}{2} \quad ; \quad h_5 = -\frac{\nu}{2} \quad ; \quad h_6 = \frac{1}{2}. \tag{2.116}$$

In different theories of gravity, these coefficients will take a different value. Thus, one can design tests of General Relativity by observationally measuring these coefficients, *e.g.* in solar system tests. We will keep them arbitrary to show that the EOB formalism works for more general theories.

We will convert $H_{1PN}[Q, P]$ to $H_{1PN}[Q, p]$, using $P[Q, p]$. This is an exercise of substitution, and neglecting terms beyond 1PN, we obtain

$$\frac{H_{1PN}[Q,p] - M}{\mu} = \left(\frac{\hat{\mathcal{P}}^2}{2} - \frac{h_N}{\hat{R}}\right) + h_1\hat{\mathcal{P}}^4 + (h_2 - \alpha_1)\,\hat{p}_r^2\hat{\mathcal{P}}^2 + (h_3 + 2\alpha_1 + \beta_1)\,\hat{p}_r^4 + h_4\frac{\hat{\mathcal{P}}^2}{\hat{R}} + h_5\frac{\hat{p}_r^2}{\hat{R}} + \frac{h_6}{\hat{R}^2}, \tag{2.117}$$

with $\hat{\mathcal{P}} = \hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{R}^2}$. Regarding the one-body Hamiltonian, we found

$$\frac{H_e[q,p]}{\mu} = 1 + \left(\frac{\hat{p}^2}{2} + \frac{a_1}{2\hat{r}}\right) - \frac{\hat{p}^4}{8} + a_1\frac{\hat{p}^2}{4\hat{r}^2} - b_1\frac{\hat{p}_r^2}{2\hat{r}} + \frac{a_2 - a_1^2/4}{2\hat{r}^2}. \tag{2.118}$$

We will convert $H_e[q, p]$ to $H_e[Q, p]$ using $q[Q, p]$. This is again an exercise of substituting and neglecting terms beyond 1PN. We obtain

$$\begin{aligned}\frac{H_e[Q,p]}{\mu} - 1 = &\left(\frac{\hat{\mathcal{P}}^2}{2} + \frac{a_1}{2\hat{R}}\right) - \left(\alpha_1 + \frac{1}{8}\right)\hat{\mathcal{P}}^4 - (\alpha_1 + 3\beta_1)\,\hat{p}_r^2\hat{\mathcal{P}}^2 \\ &+ (2\alpha_1 + \beta_1)\,\hat{p}_r^4 + \left(-\gamma_1 - \frac{a_1\alpha_1}{2} + \frac{a_1}{4}\right)\frac{\hat{\mathcal{P}}^2}{\hat{R}} \\ &+ \left(\gamma_1 - \frac{a_1}{2}(2\alpha_1 + 3\beta_1) - \frac{b_1}{2}\right)\frac{\hat{p}_r^2}{\hat{R}} \\ &+ \left(-\frac{a_1\gamma_1}{2} + \frac{a_2}{2} - \frac{a_1^2}{8}\right)\frac{1}{\hat{R}^2}.\end{aligned} \tag{2.119}$$

(iv) Functional relation between H_{1PN} and H_{EOB}

We are now ready to map the Hamiltonians. There is a subtlety in mapping Hamiltonians: the time can be redefined, but it has to preserve the canonical form $dt \wedge dH$ such that $dt \wedge dH = dt' \wedge dH'$. This relationship is the analogous of the canonical symplectic form $dq \wedge dp = dq' \wedge dp'$. Here, we perform an energy-dependent canonical rescaling of time:

$$dt_{1PN} = dt_e\frac{dH_e}{dH_{1PN}}. \tag{2.120}$$

In quantum mechanics, both canonical forms $dt \wedge dH$ and $dq \wedge dp$ lead after quantization to Heisenberg uncertainty relations.

This amounts to impose a functional relation

$$H_e = f_{\rm EOB}(H_{\rm 1PN}) \quad \Leftrightarrow \quad H_{\rm 1PN} = f^{-1}_{\rm EOB}(H_e). \tag{2.121}$$

The functional turns out to be given by

$$\frac{H_e}{\mu} - 1 = \frac{H_{\rm 1PN} - M}{\mu}\left[1 + \frac{\nu}{2}\left(\frac{H_{\rm 1PN} - M}{\mu}\right)\right] \tag{2.122}$$

at all PN orders. The Fichtenholz Hamiltonian $H_{\rm 1PN}$ is the real energy of the system, while H_e is the effective energy. We can invert the relationship to

$$E^2_{\rm real} = m_1^2 + m_2^2 + 2m_1 m_2 \frac{E_{\rm eff}}{\mu}. \tag{2.123}$$

This relation is quite fantastic and mysterious. It is independently realized in several systems such as in QED and in scattering, and has therefore some universality. Here, following the notations introduced above, we can also write this relationship as

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}. \tag{2.124}$$

We can then expand the EOB Hamiltonian as

$$H_{\rm EOB} = H_{\rm 1PN} + \text{subleading}$$

which is *resummed* thanks to the exact expression (2.124). At 1PN order, we can directly compare H_{1PN} to the 1PN expansion of the right-hand side of (2.124). At Newtonian order, no constraints arise. At 1PN order, the existence of an EOB Hamiltonian requires the condition

$$2h_2 + 3h_3 = 0, \tag{2.125}$$

which is obeyed by Einstein gravity. There is exactly one constraint at 1PN order because there are exactly 7 coefficients $h_N, h_1, ..., h_6$ for the two-body Hamiltonian $H_{\rm 1PN}$ which are needed to be mapped using 3 parameters α_1, β_1 and γ_1 in the canonical transformation $G(Q, P)$ to the one-body Hamiltonian that depends upon 3 coefficients a_1, a_2 and b_1 hidden in the functions $A(r)$ and $B(r)$. We therefore expect $7 - 3 - 3 = 1$ constraints on the 1PN dynamics. At higher orders, new conditions and new coefficients arise. At 2PN, the counting is $17 - 5 - 9 = 3$ constraints on the 17 h coefficients. Such constraints are obeyed by General Relativity, but also by scalar-tensor theories, which are modifications of General Relativity. As an exercise, we could use (2.122) to map the coefficients of the two systems. We find for each term:

$$
\begin{aligned}
1/\hat{R}: &\quad a_1 = -2; & \hat{p}_r^4: &\quad h_3 = 0;\\
\hat{\mathcal{P}}^4: &\quad \alpha_1 = -\nu/2; & \hat{\mathcal{P}}^2/\hat{R}: &\quad \gamma_1 = 1+\nu/2;\\
\hat{p}_r^2\hat{\mathcal{P}}^2: &\quad \beta_1 = 0; & \hat{p}_r^2/\hat{R}: &\quad b_1 = 2;\\
1/\hat{R}^2: &\quad a_2 = 0.
\end{aligned}
$$

The effective one-body metric is therefore

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2, \tag{2.126}$$

where the functions are given by

$$\begin{cases} A(r) = A + \dfrac{a_1}{\hat{r}} + \dfrac{a_2}{\hat{r}^2} = 1 - \dfrac{2M}{r},\\ B(r) = 1 + \dfrac{b_1}{\hat{r}} = 1 + \dfrac{2M}{r}. \end{cases} \tag{2.127}$$

This is exactly the 1PN expansion of a Schwarzschild metric of total mass M. At 2PN order, the effective metric already starts to differ from the Schwarzschild metric. The motion is described by a geodesic in that effective metric. At 2.5PN order, radiation-reaction sets in, and the motion is no longer geodesic. At 4PN, the equations of motion are no longer instantaneous in time. Since gravity is non-linear, the gravitational waves sent at 2.5PN order start to backreact at 1.5PN further order back to the system. This can be illustrated by the Feynman diagram Fig. 2.2.

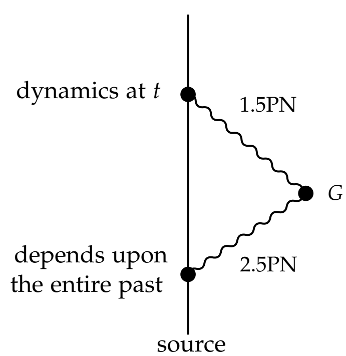


Figure 2.2: Feynmann diagram

2.3 *Gravitational self-force*

Gravitational self-force (GSF) is applicable to binaries with small mass ratios: $\varepsilon = m_2/m_1 \ll 1$. The binary's spacetime metric is expanded in powers of the mass ratio as

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \varepsilon h^{(1)}_{\alpha\beta} + \varepsilon^2 h^{(2)}_{\alpha\beta} + \ldots, \tag{2.128}$$

where $g^{(0)}_{\alpha\beta}$ describes the spacetime of the primary object, typically a Kerr black hole, but it could also be the exterior metric of a neutron star (with different multipole moments than the black hole). The corrections $h^{(n)}_{\alpha\beta}$, $n = 1, 2, \ldots$ are perturbations created by the presence of the secondary object, which could be again a black hole or another compact object. These perturbations cause a self-force on the secondary, which accelerates it away from geodesic motion. We have

$$u^\alpha \nabla_\alpha u^\mu = f^\mu[h^{(1)}, h^{(2)}, \ldots; g^{(0)}_{\alpha\beta}, u^\mu], \tag{2.129}$$

where f^μ is the self-force and is dependent on the background metric. Given that the velocity is normalized, $u^\alpha u_\alpha = -1$, the self-force is orthogonal to the velocity: $u^\mu f_\mu = 0^*$. In self-force theory, one splits the metric perturbation

* This is simply proven from $u^\alpha \nabla_\alpha \left(u^\mu u_\mu\right) = 0$ and $u^\alpha \nabla_\alpha \left(u^\mu u_\mu\right) = u_\mu u^\alpha \nabla_\alpha u^\mu = u_\mu f^\mu$.

$$h_{\alpha\beta} = \sum_{n=1}^{\infty} \varepsilon^n h^{(n)}_{\alpha\beta} \tag{2.130}$$

into a regular piece and a so-called *puncture* piece:

$$h_{\alpha\beta} = h^R_{\alpha\beta} + h^P_{\alpha\beta}, \tag{2.131}$$

where $h^R_{\alpha\beta}$ is regular at $r = 0$ and $h^P_{\alpha\beta} \sim \frac{m}{r} + \cdots + \frac{m_{\ell,m} Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}$ is the puncture piece, $Y_{\ell,m}(\theta, \phi)$ a spherical harmonic and ℓ the multipole moment expansion. The self-force acting on the secondary, per unit mass of the secondary, is given by [10]

[10] Adam Pound. Second-order gravitational self-force. *Phys. Rev. Lett.*, 109: 051101, 2012

$$f^\mu = -\frac{1}{2} P^{\mu\nu} \left(\delta^\rho_\nu - h^{R\,\rho}_{\nu}\right) \left(2\nabla^{(0)}_\alpha h^R_{\beta\rho} - \nabla^{(0)}_\rho h^R_{\alpha\beta}\right) u^\alpha u^\beta + \mathcal{O}(\varepsilon^3), \tag{2.132}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the orthogonal projector to u^μ: $P^{\mu\nu} u_\nu = 0$. The self-force at third order is unknown. Fortunately, it is not required at this time for third generation detectors. The emitted waveform is contained in the perturbations $h^{(n)}_{\alpha\beta}$, $n = 1, 2, \ldots$ Calculations involving the n^{th} perturbation are referred to as nSF.

The metric obeys Einstein's equations expanded in perturbation theory. The left-hand side of Einstein's equations is

$$G_{\mu\nu}\left[g^{(0)}_{\alpha\beta} + h_{\alpha\beta}\right] = -\frac{1}{2} E_{\mu\nu}[h_{\alpha\beta}] + \delta^2 G_{\mu\nu}[h, h] + \delta^3 G_{\mu\nu}[h, h, h] + \mathcal{O}(\varepsilon^4), \tag{2.133}$$

where $E_{\mu\nu}[h_{\alpha\beta}] = \nabla^\alpha_{(0)} \nabla^{(0)}_\alpha \overline{h}_{\mu\nu} + 2R_{(0)}{}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} \overline{h}_{\alpha\beta}$ is the linearized Einstein tensor and $\overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} g^{\mu\nu}_{(0)} h_{\mu\nu} g^{(0)}_{\alpha\beta}$ is the trace-reversed metric perturbation that obeys $\partial^\alpha \overline{h}_{\alpha\beta} = 0$ (called the harmonic/de Donder gauge). The explicit expressions for $\delta^2 G_{\mu\nu}$ and $\delta^3 G_{\mu\nu}$ are given in the *xAct* package *xPert*[11].

[11] http://www.xact.es/xPert/

The right-hand side is given by the point-particle stress-energy tensor

$$T_{\mu\nu} = m_2 \int d\tau\, u_\mu u_\nu \frac{\delta^{(4)}\left(x^\mu - z^\mu(\tau)\right)}{\sqrt{-\text{Det}\, g}} + \mathcal{O}(\varepsilon^2), \tag{2.134}$$

where $z^\mu(\tau)$ is the position of the secondary body. The second-order terms $\mathcal{O}(\varepsilon^2)$ are linear in $h^{R(1)}_{\mu\nu}$ multiplying delta functions over the worldline [12]. Waveforms have been so far generated for inspirals at 2SF order for a subset of the relevant parameter space. Bound geodesics around a Kerr black hole are triperiodic, undergoing radial, polar and azimuthal motion with frequencies Ω_r, Ω_θ and Ω_ϕ, where the values can be found in the *BlackHolePerturbation Toolkit*[13]. Over the long inspiral, the frequencies slowly evolve due to dissipation (emission of gravitational waves), while the orbital phases $\psi_A = (\psi_r, \psi_\theta, \psi_\phi)$ evolve quickly. Therefore, the perturbation can be solved using a multiscale expansion

[12] Samuel D. Upton and Adam Pound. Second-order gravitational self-force in a highly regular gauge. *Physical Review D*, 103(12), June 2021

[13] https://bhptoolkit.org/

$$h^{(n)}_{\alpha\beta} = \sum_{k^A \in \mathbb{Z}^3} h^{(n,k^A)}_{\alpha\beta}(J^{\hat{A}}, x^a) e^{ik^A \psi_A}. \tag{2.135}$$

Here, k^A are Fourier labels, $J^{\hat{A}}$ the set of phase space variables: the three frequencies $\Omega_{r,\theta,\phi}$, as well as the change of mass and spin of the central black hole over the mass ratio $\delta M, \delta J$, such that

$$M = M^{(0)} + \varepsilon\, \delta M \quad ; \quad J = J^{(0)} + \varepsilon\, \delta J, \tag{2.136}$$

and $x^a = (r, \theta, \phi)$. All time dependence is contained in J^A and ψ_A. The *fast* orbital phases evolve on the timescale

$$\psi_A \sim m_1, \tag{2.137}$$

while the *slow* phase space variables evolve on the timescale

$$J^{\hat{A}} \sim m_2 \sim \frac{m_1}{\varepsilon}. \tag{2.138}$$

The evolution of the system takes the form

$$\begin{cases} \dfrac{d\psi^A}{dt}(\varepsilon, t) = \Omega^A_{(0)}(J^{\hat{A}}) + \varepsilon\, \Omega^A_{(1)}(J^{\hat{A}}, \psi_B) + \dots \\ \dfrac{dJ_{\hat{A}}}{dt}(\varepsilon, t) = \varepsilon\, F^{(0)}_{\hat{A}}(J^{\hat{A}}, \psi_B) + \varepsilon^2 F^{(1)}_{A}(J^{\hat{A}}, \psi_B) + \dots, \end{cases} \tag{2.139}$$

where each first term corresponds to the adiabatic or 0PA order, and each second term to the post-adiabatic or 1PA order. The frequencies $\Omega^A_{(0)}(J^{\hat{A}})$ are the fundamental geodesic frequencies. The explicit ψ_B dependency of $F^{(n)}_{\hat{A}}$ and $\Omega^A_{(n)}$ can be removed with an appropriate field redefinition of the variables ψ^A and $J_{\hat{A}}$ called a *near-identity transformation* (NIT). Note that ψ^A contains the variable ϕ, but ϕ cannot appear on the right-hand side of the equations of motion because the metric is ϕ-independent, due to the axisymmetry of Kerr. Physically, $\frac{d\psi_\phi/dt}{d\psi_r/dt}$ represents the precession of the periaxis, while $\frac{d\psi_\phi/dt}{d\psi_\theta/dt}$ is the precession of the orbital angular momentum.

The system (2.139) needs to be solved simultaneously with Einstein's equations. The forcing terms of the motion depend upon the metric perturbation, and Einstein's tensor is sourced by the stress-energy tensor of the secondary. Because the time dependence is entirely encoded in $(\psi^A, J_{\hat{A}}, S_k)$, one can substitute in Einstein's equations

$$\partial_t|_{\text{fixed } r} = \frac{d\psi^A}{dt}\frac{\partial}{\partial\psi^A} + \frac{dJ_{\hat{A}}}{dt}\frac{\partial}{\partial J_{\hat{A}}}. \tag{2.140}$$

The trace-reversed metric perturbation $\bar{h}^{(n)}_{\alpha\beta}$ are typically expanded in a basis of tensorial harmonics $Y^{i\ell m}_{\alpha\beta}(r,\theta,\phi)$ where $i = 1,\ldots,10$ labels the basis, and ℓm labels the spherical harmonic decomposition. Current numerical codes include modes up to $\ell = 30$, which achieves nearly machine precision. Therefore, the trace-reversed metric perturbation is decomposed as

$$\bar{h}_{\mu\nu}(\varepsilon,t,x^i) = \sum_{n\geq 1}\varepsilon^n \sum_{\substack{i\ell m\\ k^A\in\mathbb{Z}^2}} R^n_{i\ell m}\left(J_{\hat{A}}(\varepsilon,t),r\right) e^{ik^A\psi_A(\varepsilon,t)} Y^{i\ell m}_{\mu\nu}(r,\theta,\phi). \tag{2.141}$$

Einstein's equations amount to solve for the functions $R^n_{i\ell m}\left(J_{\hat{A}},r\right)$. The multiscale structure of the equations allows to solve them in two steps, the so-called *offline* and *online* steps, which is numerically efficient for waveform generation.

In the offline step, $R^n_{i\ell m}\left(J_{\hat{A}},r\right)$ are computed for a grid of phase space parameters $J_{\hat{A}}$. This allows to compute the right-hand side of the equations of motion $\Omega^A_{(n)}(J^{\hat{A}})$ and $F^{(n)}_{\hat{A}}(J^{\hat{A}})$. All the generated data is stored.

In the online step, the equations of motion are solved (2.139) and the metric perturbation is evaluated as a time evolution, which produces the waveform. This step only amount to solve ordinary differential equations and is therefore fast to evaluate. This algorithm has been implemented in the *Fast EMRI Waveforms* package (FEW package) at 0PA (adiabatic) order, and privately at 1PA order.

The phases obey the equation

$$\frac{d\psi^A}{dt} = \Omega^A_{(0)}\left(J^{\hat{A}}(\varepsilon t)\right) + \ldots, \tag{2.142}$$

or equivalently

$$\frac{d\psi^A}{d(\varepsilon t)} = \frac{1}{\varepsilon}\Omega^A_{(0)}\left(J^{\hat{A}}(\varepsilon t)\right) + \ldots \tag{2.143}$$

with solution

$$\psi^A(t) = \frac{1}{\varepsilon}\psi^A_{(0)}(\varepsilon t) + \psi^A_{(1)}(\varepsilon t) + \mathcal{O}(\varepsilon), \tag{2.144}$$

with $\mathcal{O}(\varepsilon) \ll 1$ for $\varepsilon \ll 1$. In the case of extreme mass ratio inspirals (EMRIs) to be detected by LISA, the waveform needs to be accurate over ~ 1 radian over the entire inspiral. Therefore, 1PA order, *i.e.* solving for $\psi^A_{(0)}(\varepsilon t)$ and $\psi^A_{(1)}(\varepsilon t)$ provide accurate enough waveforms for future data analysis with LISA.

Caveat 1

The integer power ε ansatz used to derive this multiscale expansion is only valid as long as $k^A \Omega_A \neq 0$. In the latter case, there is a *resonance* and the consistent evolution is in half-integer powers of ε. In that case, one needs the 0PA, 0.5PA and 1PA contributions. Essentially all LISA-type EMRIs pass through at least one dynamically significant resonance.

Caveat 2

The inspiral timescale expansion breaks down as the secondary object approaches in phase space the separatrix between bound and plunging orbits. In the case of a quasi-circular orbit around Schwarzschild, the separatrix is the *innermost stable circular orbit* (ISCO), at radius $r_{\text{ISCO}} = 6M$ beyond which no stable circular orbit exists. In the approach to the separatrix, one starts a different transition-to-plunge regime with typical evolution timescale $\varepsilon^{1/5} t$, as found by Ori-Thorne [14] and Buonanno-Damour [15]. This regime is also a specialty of Brussels. The waveforms are then generated using an asymptotically matched expansion scheme. The waveforms for a quasi-circular inspiral, transition-to-plunge and merger look like Fig. 2.3.

[14] Amos Ori and Kip S. Thorne. The Transition from inspiral to plunge for a compact body in a circular equatorial orbit around a massive, spinning black hole. *Phys. Rev. D*, 62:124022, 2000

[15] Alessandra Buonanno and Thibault Damour. Transition from inspiral to plunge in binary black hole coalescences. *Phys. Rev. D*, 62:064015, 2000

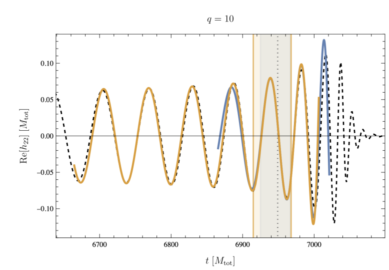


Figure 2.3: Waveforms of quasi-circular inspiral and coalescence for a mass ratio $q = 10$. Self-force models (solid lines) are compared to a SXS numerical simulation (dashed lines). From L. Küchler, G. Compère, L. Durkan and A. Pound. *SciPost Selections Physics*, 2024.

Caveat 3

Right after the leading effects due to masses of the bodies, the spin effects are important. The primary spin effects are included by considering a Kerr black hole background. The secondary spin effects are included by considering the stress-energy tensor of spinning particle. The theory of a relativistic spinning particle is called the *Mathisson-Papapetrou* theory. Including higher multipole moments is possible, which is described by Dixon's theory formulated in the 1970's. In the Mathisson-Papapetrou-Dixon (MPD) theory, the particle motion is described by the position $z^\mu(t)$, the velocity $v^\mu = dz^\mu/d\tau$, the momentum $p^\mu(\tau)$ which is not necessarily aligned with the velocity and the spin tensor $S^{\mu\nu}(\tau)$. It is necessary to fix the origin inside the body which leads to an algebraic condition. The most common choice is the so-called *Tulczyjew condition* $S^{\mu\nu} p_\nu = 0$. This allows to define the spin vector

$$S_\beta = \frac{1}{2\mu} \varepsilon_{\beta\mu\nu\alpha} S^{\mu\nu} p^\alpha, \tag{2.145}$$

where $\mu = \sqrt{-p_\mu p^\mu}$. The spin length or magnitude is

$$S^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = S^\mu S_\mu. \tag{2.146}$$

Then, the MPD equations, which are the equations of motion, are given by

$$\begin{cases} \frac{D}{D\tau} p^\mu = \frac{1}{2} S^{\alpha\beta} R_{\alpha\beta\nu}{}^{\mu} v^\nu + F^\mu_{\text{force}} \\ \frac{D}{D\tau} S^{\mu\nu} = 2p^{[\mu} v^{\nu]} + N^{\mu\nu}_{\text{torque}}, \end{cases} \tag{2.147}$$

where F^μ_{force} and $N^{\mu\nu}_{\text{torque}}$ depend upon the higher multipole moments or the self-force effects. The condition $S^{\mu\nu} p_\nu = 0$ leads to

$$\frac{D}{D\tau} \left(S^{\mu\nu} p_\nu \right) = 0. \tag{2.148}$$

Using the equations of motion, this leads to an algeabraic constraint that allows to solve for v^α in terms of p^α:

$$v^\mu = \frac{p^\mu}{\mu} + \frac{1}{2\mu^2} S^{\mu\nu} p^\rho R_{\nu\rho\alpha\beta} S^{\alpha\beta} + \mathcal{O}(S^3). \tag{2.149}$$

In the absence of sources, $F^\mu_{\text{force}} = 0$ and $N^{\mu\nu}_{\text{torque}} = 0$, the spin magnitude S and the mass μ are conserved along the motion.

Let us now discuss how to solve the motion of a secondary body around a primary body. Given the coupling between the two bodies, it also involves solving Einstein's equations. At adiabatic order, the inspiral evolution is driven by the fluxes of energy, angular momentum and *Carter's constant*, which are equivalent to the fluxes of the fundamental frequencies Ω_r, Ω_θ and Ω_ϕ. The Kerr black hole admits a non-trivial Killing tensor $K_{\mu\nu}$ and its associated conserved quantity $K = K_{\mu\nu} v^\mu v^\nu$ along the geodesic motion. We can check that

$$\frac{D}{D\tau} K = \frac{DK_{\mu\nu}}{D\tau} v^\mu v^\nu + 2K_{\mu\nu} \frac{Dv^\mu}{D\tau} v^\nu = 0, \tag{2.151}$$

because $\frac{Dv^\mu}{D\tau} = 0$ by the geodesic equation and $v^\mu \frac{DK_{\mu\nu}}{D\tau} = v^\mu v^\alpha \nabla_\alpha K_{\mu\nu} = v^\mu v^\alpha \nabla_{(\alpha} K_{\mu)\nu} = 0$ after symmetrization over $\alpha\mu$ and by definition of a Killing tensor. The existence of the four conserved quantities: m_2, $E = -(\partial_t)^\mu g_{\mu\nu} v^\nu$, $J = (\partial_\phi)^\mu g_{\mu\nu} v^\nu$ and K makes the geodesic motion integrable: the Poisson bracket between the conserved quantities is zero. At adiabatic order, we can therefore only limit ourselves to compute the fluxes of E, J and Q to drive the inspiral. In particular, we do not need all metric perturbations. It turns out that one can built from the metric perturbation around Kerr a complex scalar called $\delta\psi_4$ which encodes all the information on gravitational wave propagation and which moreover allows to separate the polar and radial motions. It was found by Saul Teukolsky in 1973 while he was accomplishing his PhD under the supervision of Kip Thorne.

A Killing tensor $K_{\mu\nu}$ is a symmetric tensor which obeys $\nabla_{(\mu} K_{\nu)\alpha} = 0$. The Carter constant K is a fourth conserved geodesic quantity which makes the geodesic motion around Kerr integrable in the sense of Liouville. It is built as

$$K = K_{\mu\nu} v^\mu v^\nu, \tag{2.150}$$

where $K_{\mu\nu}$ is a Killing tensor and v^μ the velocity. A trivial Killing tensor can be written as a sum of terms that are the direct product of 2 Killing vectors: $K_{\mu\nu} = \xi^{(1)}_{(\mu} \xi^{(2)}_{\nu)}$. A non-trivial Killing tensor cannot be written in that form.

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