PHYS-F-417

Introduction to Supersymmetry - Exercises

1. **BPS supermultiplets**

   Derive the representations of the superalgebra with $\mathcal{N} > 1$ supersymmetries and central extensions:

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^{IJ}, \\ \{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ}, \qquad Z^{IJ} = -Z^{JI}. \end{aligned}$$

   It is best to choose a particular (simple) form for the central charge Z^{IJ} matrix (block diagonal with entries given by the eigenvalues).

   Show that there is a lower bound on the mass. Then, show that the "length" of the supermultiplets varies if there are some relations between the mass and the eigenvalues of the central charge matrix.

   For a reference, see A. Bilal, arXiv:hep-th/0101055, section 3.4.

   [The name BPS comes from Bogomol'nyi-Prasad-Sommerfield and the theory of monopoles, in which relations between the mass and the charge also appear. Such BPS objects and the shortening of their supermultiplets are extremely important in establishing non-perturbative dualities in string theory.]

2. **The supercurrent: a Noether current for supersymmetry**

   Consider the Wess-Zumino model for a scalar multiplet, in components. Find the Noether current associated to supersymmetry transformations, both for the free and for the interacting theory (for the latter, one may use the fact that interaction and mass terms descend from a superpotential).

   Recall that the definition of the Noether current is:

$$\epsilon J^\mu + \bar\epsilon \bar{J}^\mu = \sum_{X=\phi,\psi,f} \frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \delta X - V^\mu,$$

   where V^μ is such that upon a supersymmetry transformation,

$$\delta \mathcal{L} = \partial_\mu V^\mu.$$

   For a reference, see e.g. S. P. Martin, "A Supersymmetry primer," arXiv:hep-ph/9709356, section 3.

   [If one is left hungry after this exercise, she/he may continue by applying the Noether method, leading to the coupling to the gravitino and eventually to supergravity. It can then be shown that the supercurrent itself transforms under supersymmetry into the energy-momentum tensor $T^{\mu\nu}$ (which couples to the graviton).]

3. **Non-linear Sigma models and Kähler potential**

   Consider the manifestly supersymmetric action

$$\int d^2\theta d^2\bar{\theta}\; K(\Phi_i, \bar{\Phi}_{\bar{\jmath}}) + \int d^2\theta\; W(\Phi_i) + c.c.,$$

   with $K(z_i, z^*_{\bar{\jmath}})$ a real function of N complex variables z_i. Taking

$$\Phi_i = \phi_i + \sqrt{2}\theta\psi_i + \theta^2 f_i,$$

   develop the action in components, also solving in the process the equations of motion for the auxiliary fields.

   For a reference, see A. Bilal, arXiv:hep-th/0101055, section 7.1.

   [This kind of models, which are not necessarily renormalizable, arises when considering effective theories, i.e. theories which are valid only for energies lower than a certain scale. The effects of the new physics that kicks in at higher energies are encoded in the non-linearities of the effective theory. What is shown here is that the "target space" has the geometry of a Kähler manifold due to supersymmetry.]

4. **Moduli space of SQCD**

   Describe the moduli space of $SU(N_c)$ SQCD with N_f flavors, as N_f is varied with respect to N_c. In particular:

   (i) Give the most general solution (up to gauge and global symmetry rotations) of the D-flatness equations for $N_f < N_c$ and for $N_f \geq N_c$.

   (ii) For the gauge invariant description of the moduli space, find the chiral gauge invariant operators in the cases $N_f < N_c$, $N_f = N_c$, $N_f = N_c + 1$ and $N_f > N_c + 1$. Find in which cases the operators satisfy relations among them. Check against the expected dimension of the moduli space.

   For a reference, see e.g. M. A. Shifman, "Nonperturbative dynamics in supersymmetric gauge theories," Prog. Part. Nucl. Phys. **39** (1997) 1 [arXiv:hep-th/9704114], section 3.1.

   [These classical moduli spaces have non-trivial quantum corrections when the strongly coupled dynamics of SQCD is taken into account.]

5. **Supertrace theorem for a SUSY breaking non-linear Sigma model**

Consider a model like those in Exercise 3, which breaks SUSY spontaneously (i.e. one cannot set $W_i = 0$ for all i at the same time). Compute the difference of squared masses of bosons and fermions around the SUSY breaking vacuum in this more general set up and verify that now the supertrace no longer vanishes, but is rather determined by the non-trivial Kähler potential.

[This should make it obvious why the supertrace theorem only rules out tree-level (classical) spontaneous SUSY breaking in the SSM, but does not affect other scenarios based on radiative corrections and/or supergravity.]

6. **A minimal model of gauge mediation of SUSY breaking**

We consider a theory with a "visible" sector (mimicking the MSSM) composed of massless SQED:

$$\mathcal{L}_{vis} = \int d^2\theta d^2\bar{\theta}\,(\bar{Q}e^{2gV}Q + \tilde{Q}e^{-2gV}\bar{\tilde{Q}}) - \frac{1}{4}\int d^2\theta\,\mathcal{W}^\alpha\mathcal{W}_\alpha + c.c.,$$

and a "messenger" sector coupled to a spurion

$$\mathcal{L}_{mess} = \int d^2\theta d^2\bar{\theta}\,(\bar{\Phi}e^{2gV}\Phi + \tilde{\Phi}e^{-2gV}\bar{\tilde{\Phi}}) + \int d^2\theta\,X\Phi\tilde{\Phi} + c.c.$$

The spurion has a VEV given by

$$X = M + \theta^2 F.$$

It thus summarizes the SUSY breaking taking place in a "hidden" sector, which we assume to be responsible for the value of F.

In this model, show that:

(i) The messengers have a split spectrum at tree level with a vanishing supertrace.

(ii) The gaugino acquires a mass at one-loop level. Write the Feynman diagram that is responsible for it and compute it. (It is best done using Feynman rules for Weyl fermions. These are easily derived from the path integral.)

(ii) The sfermions (the scalar components of Q and $\tilde{Q}$) acquire a mass at two-loops. Write the Feynman diagrams contributing to it. Compute the overall contribution to the mass (paying attention for instance to the sign!). Is the supertrace vanishing also in the visible matter sector?

Observe the pattern of visible sector soft masses thus generated.

For a reference, see e.g. P. Meade, N. Seiberg and D. Shih, "General Gauge Mediation," Prog. Theor. Phys. Suppl. **177** (2009) 143 [arXiv:0801.3278 [hep-ph]], section 3 and appendix A.

[This simple model of mediation of SUSY breaking is called "gauge mediation" because the gauge multiplet of the visible sector is the first one to acquire non-SUSY masses by radiative corrections. It then generates non-SUSY masses in the matter sector by further radiative corrections. This model is one of the most popular ones and can be generalized in many ways.]

7. **The Goldstino from the supercurrent, and its effective Lagrangian**

Explore the relation between the Goldstino, the massless fermion related to spontaneous breaking of SUSY, to the supercurrent, in a way similar to what is done for the Goldstone boson, as for instance in J. Goldstone, A. Salam and S. Weinberg, "Broken Symmetries," Phys. Rev. **127** (1962) 965.

This approach shows that the existence and the masslessness of the Goldstino is independent on the classical Lagrangian description of the SUSY breaking, and is thus applicable also to strongly coupled situations.

The Goldstino effective Lagrangian and its couplings to matter can be determined universally through this approach (again, see e.g. S. P. Martin, "A Supersymmetry primer," arXiv:hep-ph/9709356, section 7.5).

Note that the linear Goldstino couplings to matter can be presented either in derivative or, after using the equations of motion (or equivalently, field redefinitions), in non-derivative form. In the latter form, they are explicitly proportional to the soft masses that are present in the SUSY breaking vacuum. Show that the same couplings can be derived by promoting to a dynamical superfield X the spurion encoding the soft terms, in such a way that

$$X = \cdots + \sqrt{2}\theta G + \theta^2 F$$

where G is the Goldstino and F the SUSY breaking VEV. (For a recent discussion of the Goldstino superfield, see Z. Komargodski and N. Seiberg, "From Linear SUSY to Constrained Superfields," JHEP **0909** (2009) 066 [arXiv:0907.2441 [hep-th]].)